IRSAF

Istituto di Ricerca Scientifica e di Alta Formazione

Working Papers in Economics and Finance

Consumption sequences and time preference

Pietro Senesi

November, 2023

ISBN 978-88-6991-160-6

Consumption sequences and time preference

Pietro Senesi*

November, 2023

Abstract

Time preference is a fundamental parameter in models of intertemporal consumption choice. The preference for advancing the timing of consumption studied by Böhm-Bawerk[2] was referred to as *time preference* by Fisher[4], and *impatience* by Fisher[5]. Time preference was given a precise definition by Koopmans[6] in relation to sets of countable (future) consumption sequences and classes of ordinal utility functions consistent with preferences structures that order consumption programs according to criteria that are independent of the passage of time (time traslations of consumption programs). Specifications of the latter approach are e.g. Uzawa[7] and Epstein and Hynes[3]. This paper studies the dependency of endogenous time preference on consumption sequences in a discrete time formulation of Epstein and Hynes[3].

Journal of Economic Literature Classification Numbers: D91, D90, D81, D11, D12.

*Department of Human and Social Sciences, University of Naples "L'Orientale", Largo S. Giovanni Maggiore, 30 - 80134 Naples, Italy; email: psenesi@unior.it, Phone: +39 081 6909482, Fax: +39 081 6909442

1 Introduction

Time preference is a fundamental parameter in models of intertemporal consumption choice. The preference for advancing the timing of consumption studied by Böhm-Bawerk[2] was referred to as *time preference* by Fisher[4], and *impatience* by Fisher[5]. Time preference was given a precise definition by Koopmans[6] in relation to sets of countable (future) consumption sequences and classes of ordinal utility functions consistent with preferences structures that order consumption programs according to criteria that are independent of the passage of time (time traslations of consumption programs). Specifications of the latter approach are e.g. Uzawa[7] and Epstein and Hynes[3][1].
This paper studies the dependency of endogenous time preference on consumption sequences in a discrete time formulation of Epstein and Hynes[3].
The plan of the paper is as follows: Section 2 reviews the Epstein and Hynes [3] model of endogenous time preference framed in continuous time. Section 3 extends such model to discrete time, while Section 4 studies intertemporal substitution between consumption at different times and derives closed form expressions for both the time preference rate and the utility discount rate. Section 5 ends the paper with a summary of results and conclusions.

2 Continuous time formulation

In the continuous time framework of Epstein and Hynes[3], with $T \in \mathbb{R}^+$, agents order consumption sequences according to the following criterion

$$U({}_0C_\infty) = -\int_0^\infty e^{-\int_0^t u(c)d\tau} dt \tag{1}$$

which allows for additive separability of preferences at any time $T \in [0, \infty]$

$$G(T, C_T, \phi) = -\int_0^T e^{-\int_0^t u(c)d\tau} dt + \phi(T) \cdot e^{-\int_0^T u(c)dt} \tag{2}$$

with discounted aggregate utility from future consumption denoted by

$$U[{}_TC_\infty] = -\int_T^\infty e^{-\int_T^t u(c)d\tau} dt \cdot e^{-\int_0^T u(c)dt} \tag{3}$$

and undiscounted aggregate utility from future consumption denoted by

$$\phi(T) = -\int_T^\infty e^{-\int_T^t u(c)d\tau} dt \tag{4}$$

additively aggregated to utility from past consumption, upon discounting according to the factor

[1]Another approach is by Becker and Mulligan[1] who assume consumers spend resources on imagining future pleasures to perceive them as less remote, thereby making impatience endogenous.

$$e^{-\int_0^T u(c)dt} \tag{5}$$

Thus, eq.(1) can be rewritten as

$$G(T, C_T, \phi) = -\int_0^T e^{-\int_0^t u(c)d\tau} dt + \left[-\int_T^\infty e^{-\int_T^t u(c)d\tau} dt\right] \cdot e^{-\int_0^T u(c)dt} \tag{6}$$

with the expression (5) acting like a discount factor over the time interval $[0, T]$, i.e. in order to aggregate the utility flow $[-\int_T^\infty e^{-\int_T^t u(c)d\tau} dt]$, it has to be discounted by (5).

Given the additive structure of (6), the marginal utility of a consumption increment at time T is

$$U_T({}_TC_\infty) = u'(c(T)) \cdot \left\{\int_T^\infty e^{-\int_T^t u(c(\tau))}\right\} \cdot e^{-\int_0^T u(c)dt} \tag{7}$$

which in Epstein and Hynes[3] is written more compactly as

$$U_T({}_TC_\infty) = u'(c(T)) \cdot \left\{\int_T^\infty e^{-\int_0^t u(c(\tau))}\right\} \tag{8}$$

Its rate of change at time T, along a sequence of constant consumption levels, can be computed as the (negative of) the derivative (with respect to T) of the logarithm of (7), and is a measure of the pure rate of time preference, denoted by

$$\rho = -\frac{d}{dT}\{log[U_T({}_TC_\infty)]\} = \left\{\int_T^\infty e^{-\int_0^t u(c(\tau))}\right\}^{-1} = -[U_T({}_TC_\infty)]^{-1} \tag{9}$$

3 Discrete time formulation

When $T \in \mathbb{N}$, agents order consumption sequences according to the following criterion

$$U({}_1C_\infty) = -\sum_{t=1}^{\infty} e^{-\sum_{\tau=1}^t u(c(\tau))} \tag{10}$$

which allows for additive separability of preferences at any time T

$$U({}_1C_\infty) = -\sum_{t=1}^{T} e^{-\sum_{\tau=1}^t u(c(\tau))} + \left\{-\sum_{t=T+1}^{\infty} e^{-\sum_{\tau=1}^t u(c(\tau))}\right\} \tag{11}$$

$$U({}_1C_\infty) = -\sum_{t=1}^{T} e^{-\sum_{\tau=1}^t u(c(\tau))} + U({}_{T+1}C_\infty) \tag{12}$$

$$U({}_1C_\infty) = U({}_1C_T) + U({}_{T+1}C_\infty) \tag{13}$$

$$U({}_1C_\infty) = -\sum_{t=1}^{T} e^{-\sum_{\tau=1}^{t} u(c(\tau))} + \{e^{-\sum_{t=1}^{T} u(c(t))}\} \cdot \left\{ -\sum_{t=T+1}^{\infty} e^{-\sum_{\tau=T+1}^{t} u(c(\tau))} \right\} \tag{14}$$

$$U({}_1C_\infty) = U({}_1C_T) + \{e^{-\sum_{t=1}^{T} u(c(t))}\} \cdot \phi(T+1) \tag{15}$$

$$U({}_1C_\infty) = -\sum_{t=1}^{T} e^{-\sum_{\tau=1}^{t} u(c(\tau))} + \{e^{-\sum_{t=1}^{T} u(c(t))}\} \cdot \phi(T+1) \tag{16}$$

with undiscounted aggregate utility from future consumption denoted by

$$\phi(T+1) = -\sum_{t=T+1}^{\infty} e^{-\sum_{\tau=T+1}^{t} u(c(\tau))} \tag{17}$$

and

$$e^{-\sum_{t=1}^{T} u(c(t))} \tag{18}$$

a factor depending on consumption history up to time T, that discounts the utility of consumption sequences enjoyed from $T+1$ to ∞. Such discount factor is distinct from the rate of impatience, and increases with T as longer consumption histories create the habit of discounting more heavily the utility enjoyed from future consumption sequences.

Given the additive structure (16), the marginal utility of a consumption increment at time T is

$$U_T({}_1C_\infty) = U_T({}_TC_\infty) = u'(c(T)) \cdot \left\{ \sum_{t=T}^{\infty} e^{-\sum_{\tau=1}^{t} u(c(\tau))} \right\} \tag{19}$$

$$U_T({}_1C_\infty) = u'(c(T)) \cdot \{e^{-\sum_{t=1}^{T-1} u(c(t))}\} \cdot \left\{ \sum_{t=T}^{\infty} e^{-\sum_{\tau=T}^{t} u(c(\tau))} \right\} \tag{20}$$

or

$$U_T({}_1C_\infty) = u'(c(T)) \cdot \left\{ \sum_{t=T}^{\infty} e^{-\sum_{\tau=1}^{t} u(c(\tau))} \right\} \tag{21}$$

and its rate of change at time T, along a sequence of constant consumption levels, can be computed as

$$\rho(T) = \left\{ \sum_{t=T+1}^{\infty} e^{-\sum_{\tau=1}^{t} u(c(\tau))} \right\}^{-1} = -[U({}_{T+1}C_\infty)]^{-1} \tag{22}$$

or

$$\rho(T) = \left\{ [e^{-\sum_{t=1}^{T} u(c(t))}] \cdot \left[\sum_{t=T+1}^{\infty} e^{-\sum_{\tau=T+1}^{t} u(c(\tau))} \right] \right\}^{-1} \tag{23}$$

$$= -\{[e^{-\sum_{t=1}^{T} u(c(t))}] \cdot \phi({}_{T+1}C_{\infty})\}^{-1} \tag{24}$$

Hence, the impatience rate is equal to the reciprocal of the opposite of utility from the stream of future consumption discounted by an amount depending from past consumption habits. The smaller T, the lesser discounting from past consumption habits and the larger the stream of future consumption levels. Thus, the rate of impatience is lower for younger agents.

The marginal utility of a consumption increment at time $T+1$ is

$$U_{T+1}({}_1C_{\infty}) = u'(c(T+1)) \cdot \{e^{-\sum_{t=1}^{T} u(c(t))}\} \cdot \left\{ \sum_{t=T+1}^{\infty} e^{-\sum_{\tau=T+1}^{t} u(c(\tau))} \right\} \tag{25}$$

4 Intertemporal substitution and time preference

In this Section the relationship between consumption sequences and marginal rates of substitution in consumption at different times is examined to the end of characterizing an endogenous time preference rate in a closed form expression. In particular, a rate of time preference is obtained as the percentage change in marginal utility due purely to the passing of time across two subsequent periods. This is here illustrated in a setting with $T \in 1, \ldots, \overline{T} < \infty$, more suitable to study both the quantitative and empirical relationship between consumption sequences and impatience, and the dependency of the rate of time preference on the agent's lifetime duration. If $\overline{T} = 2$

$$U({}_1C_2) = -e^{-[u(c(1))]} - e^{-[u(c(1))+u(c(2))]} \tag{26}$$

$$= U({}_1C_1) + U({}_2C_2) \tag{27}$$

$$= -e^{-[u(c(1))]} + e^{-u(c(1))} \cdot \{-e^{-u(c(2))}\} \tag{28}$$

$$= -e^{-[u(c(1))]} + e^{-u(c(1))} \cdot \phi(2)_2 \tag{29}$$

where $\phi(2)_2 = \phi(2)_{\overline{T}} \neq U({}_2C_2)$ is undiscounted aggregate sub-utility from consumption beyond time 1 and until time $\overline{T}$[2]

$$U({}_1C_2) = -\sum_{t=1}^{2} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]} \tag{30}$$

[2]Please note that, if $e^{-u(c(1))}$ is interpreted as a discount factor, then a familiar recursive expression of the type $U(c_1) + \delta(1) \cdot U(c_2))$ obtains with $\delta(1) = e^{-u(c(1))}$, $\delta(t) = \{e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}\}^{(\min\{t-1,1\})}$ and $t \in \{1, \ldots, \infty\}$. This would be a specification of intertemporal preferences with habit discounting of future utilities.

$$U({}_2C_2) = -\sum_{t=2}^{2} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]} = -e^{-[u(c(1))+u(c(2))]} \tag{31}$$

The marginal utility of a consumption increase at time $T = 1$ is

$$\begin{aligned} U_1({}_1C_2) &= u'(c(1)) \cdot [-e^{-[u(c(1))]}] + u'(c(1)) \cdot [-e^{-[u(c(1))+u(c(2))]}] \\ &= u'(c(1)) \cdot [-e^{-[u(c(1))]} - e^{-[u(c(1))+u(c(2))]}] \\ &= u'(c(1)) \cdot [-\sum_{t=1}^{2} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}] \\ &= u'(c(1)) \cdot U({}_1C_2) \end{aligned}$$

The marginal utility of a consumption increase at time $T = 2$ is

$$\begin{aligned} U_2({}_1C_2) &= u'(c(2)) \cdot [-e^{-[u(c(1))+u(c(2))]}] \\ &= u'(c(2)) \cdot [-e^{-[\sum_{\tau=1}^{2} u(c(\tau))]}] \\ &= u'(c(2)) \cdot [-\sum_{t=2}^{2} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}] \\ &= u'(c(2)) \cdot U({}_2C_2) \end{aligned}$$

The MRS between consumption at time $T = 1$ and consumption at time $T = 2$ is

$$\begin{aligned} &\frac{\Delta c_2}{\Delta c_1} \\ &= -\frac{U_1({}_1C_2)}{U_2({}_1C_2)} \\ &= -\frac{u'(c(1)) \cdot [-e^{-[u(c(1))]}] + u'(c(1)) \cdot [-e^{-[u(c(1))+u(c(2))]}]}{u'(c(2)) \cdot [-e^{-[u(c(1))+u(c(2))]}]} \\ &= -\frac{u'(c(1))}{u'(c(2))} \cdot [1 + \frac{e^{-[u(c(1))]}}{e^{-[u(c(1))+u(c(2))]}}] \\ &= -\frac{u'(c(1))}{u'(c(2))} \cdot [1 + \frac{e^{-[u(c(1))]}}{e^{-u(c(1))} \cdot e^{-u(c(2))]}}] \\ &= -\frac{u'(c(1))}{u'(c(2))} \cdot [1 + \frac{1}{e^{-[u(c(2))]}}] \end{aligned}$$

The local (i.e. between times 1 and 2) rate of time preference is the percentage change in marginal utility due purely to the passing of time from $T = 1$ to $T = 2$, at constant consumption, i.e. $c(1) = c(2)$. Such percentage change can be calculated as

$$\begin{aligned}\rho_{1,2}(2) = \frac{U_1({}_1C_2) - U_2({}_1C_2)}{U_2({}_1C_2)} \\ = \frac{U_1({}_1C_2)}{U_2({}_1C_2)} - 1 \\ = 1 + \frac{1}{e^{-[u(c(2))]}} - 1 \\ = \frac{1}{e^{-[u(c(2))]}} \\ = -\frac{1}{\phi(2)_2}\end{aligned}$$

If $\overline{T} = 3$

$$U({}_1C_3) = U({}_1C_2) + [-e^{-[u(c(1))+u(c(2))+u(c(3))]}] \tag{32}$$

$$= U({}_1C_2) + e^{-[u(c(1))]} \cdot U({}_2C_3) \tag{33}$$

$$= e^0 \cdot \{-e^{-u(c(1))}\} + e^{-u(c(1))} \cdot \{-e^{-u(c(2))}\} \tag{34}$$

$$+e^{-[u(c(1))+u(c(2))]} \cdot \{-e^{-u(c(3))}\} \tag{35}$$

More compactly, aggregate intertemporal utility is[3]

$$U({}_1C_3) = -\sum_{t=1}^{3} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]} \tag{37}$$

The marginal utility of a consumption increase at time $T = 1$ is

$$\begin{aligned}U_1({}_1C_3) = u'(c(1)) \cdot [-e^{-[u(c(1))]}] + u'(c(1)) \cdot [-e^{-[u(c(1))+u(c(2))]}] \\ +u'(c(1)) \cdot [-e^{-[u(c(1))+u(c(2))+u(c(3))]}] \\ = u'(c(1)) \cdot [-\sum_{t=1}^{3} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}] \\ = u'(c(1)) \cdot [-e^{-[u(c(1))]}] + u'(c(1)) \cdot [-\sum_{t=2}^{3} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}] \\ = u'(c(1)) \cdot U({}_1C_3)\end{aligned}$$

Since

$$\sum_{t=1}^{3} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]} < \sum_{t=1}^{2} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]} \tag{38}$$

[3] The maximand in equation (37) can be written in vector product notation, i.e.

$$[e^0, e^{-[u(c(1))]}, e^{-[u(c(1))+u(c(2))]}][-e^{-[u(c(1))]}, -e^{-[u(c(2))]}, -e^{-[u(c(3))]}]' \tag{36}$$

.

a longer sequence of future consumption levels (entailed e.g. by a greater life expectancy) decreases the marginal utility of consumption at time $T=1$.

The marginal utility of a consumption increase at time $T=2$ is

$$U_2({}_1C_3) = u'(c(2)) \cdot [-e^{-[u(c(1))+u(c(2))]}] + u'(c(2)) \cdot [-e^{-[u(c(1))+u(c(2))+u(c(3))]}]$$
$$= u'(c(2)) \cdot [-\sum_{t=2}^{3} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}]$$

and the marginal utility of a consumption increase at time $T=3$ is

$$U_3({}_1C_3) = u'(c(3)) \cdot [-e^{-[u(c(1))+u(c(2))+u(c(3))]}]$$
$$= u'(c(3)) \cdot [-e^{-[\sum_{\tau=1}^{3} u(c(\tau))]}]$$
$$= u'(c(3)) \cdot [-\sum_{t=3}^{3} [e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}]]$$

The MRS between consumption at time $T=1$ and consumption at time $T=2$ becomes

$$\frac{\Delta c_2}{\Delta c_1} = -\frac{U_1({}_1C_2)}{U_2({}_1C_2)}$$
$$= -\frac{u'(c(1)) \cdot [e^{-[u(c(1))]}] + u'(c(1)) \cdot [e^{-[u(c(1))+u(c(2))]}] + u'(c(1)) \cdot [e^{-[u(c(1))+u(c(2))+u(c(3))]}]}{u'(c(2)) \cdot [e^{-[u(c(1))+u(c(2))]}] + u'(c(2)) \cdot [e^{-[u(c(1))+u(c(2))+u(c(3))]}]}$$
$$= -\frac{u'(c(1))}{u'(c(2))} \cdot [1 + e^{-[u(c(1))]} + e^{-[u(c(1))+[u(c(2))]}]$$
$$= -\frac{u'(c(1)) \cdot [e^{-[u(c(1))]}] + u'(c(1)) \cdot \sum_{t=2}^{3} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}}{u'(c(2)) \cdot \sum_{t=2}^{3} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}}$$
$$= -\frac{u'(c(1))}{u'(c(2))} \cdot [1 + \frac{e^{-[u(c(1))]}}{\sum_{t=2}^{3} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}}]$$
$$= -\frac{u'(c(1))}{u'(c(2))} \cdot [1 + \frac{e^{-[u(c(1))]}}{e^{-[u(c(1))+u(c(2))]} + e^{-[u(c(1))+u(c(2))+u(c(3))]}}]$$
$$= -\frac{u'(c(1))}{u'(c(2))} \cdot [1 + \frac{e^{-[u(c(1))]}}{e^{-[u(c(1))]} \cdot [e^{-[u(c(2))]} + e^{-[u(c(2))+u(c(3))]}]}]$$
$$= -\frac{u'(c(1))}{u'(c(2))} \cdot [1 + \frac{1}{e^{-[u(c(2))]} + e^{-[u(c(2))+u(c(3))]}}]$$
$$= -\frac{u'(c(1))}{u'(c(2))} \cdot [1 + [\frac{1}{\phi(2)_3}]]$$
$$= -\frac{u'(c(1))}{u'(c(2))} \cdot [1 + \rho]$$

Thus, in order to maintain intertemporal utility constant, a reduction on current consumption must be compensated by an amount of additional future consumption equal to the MRS, measured with the instantaneous utility function, $[-\frac{u'(c(1))}{u'(c(2))}]$, plus a "premium" equal to the inverse of future intertemporal undiscounted sub-utility, $-\frac{1}{\phi(2)_3}$. The larger the absolute value of future sub-utility, the smaller the premium that needs to be added to the intertemporal MRS.

Such premium measures (the rate of) time preference at $T = 1$ (with a lifetime length of 3 periods)

$$\begin{aligned}\rho_{1,2}(3) &= \frac{U_1({}_1C_2) - U_2({}_1C_2)}{U_2({}_1C_2)} \\ &= \frac{U_1({}_1C_2)}{U_2({}_1C_2)} - 1 \\ &= -[\frac{1}{\phi(2)_3}]\end{aligned}$$

Since $\rho_{1,2}(3) < \rho_{1,2}(2)$, time preference monotonically decreases as the lifetime duration extends. Similarly, $\rho_{2,3} > \rho_{1,3}$, and impatience increases as the residual lifetime shrinks due to aging.

In general

$$U({}_1C_{\overline{T}}) = -e^{-[u(c(1))]} - e^{-[u(c(1))+u(c(2))]} \tag{39}$$

$$= U({}_1C_1) + U({}_2C_2) \tag{40}$$

$$= -e^{-[u(c(1))]} + e^{-u(c(1))} \cdot \{-e^{-u(c(2))}\} \tag{41}$$

$$= -e^{-[u(c(1))]} + e^{-u(c(1))} \cdot \phi(2)_2 \tag{42}$$

Finally, the MRS between consumption at time $T = 1$ and consumption at time $T = 3$ is

$$\frac{\Delta c_3}{\Delta c_1} = \frac{U_1({}_1C_2)}{U_3({}_1C_3)}$$

$$= \frac{u'(c(1)) \cdot [e^{-[u(c(1))]}] + u'(c(1)) \cdot [e^{-[u(c(1))+u(c(2))]}] + u'(c(1)) \cdot [e^{-[u(c(1))+u(c(2))+u(c(3))]}]}{u'(c(3)) \cdot [e^{-[u(c(1))+u(c(2))+u(c(3))]}]}$$

$$= \frac{u'(c(1))}{u'(c(3))} \cdot [1 + e^{-[u(c(1))]} + e^{-[u(c(1))]+[u(c(3))]}]$$

$$= \frac{u'(c(1)) \cdot [e^{-[u(c(1))]}] + u'(c(1)) \cdot \sum_{t=2}^{3} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}}{u'(c(3)) \cdot \sum_{t=2}^{3} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}}$$

$$= \frac{u'(c(1))}{u'(c(3))} \cdot [1 + \frac{e^{-[u(c(1))]}}{\sum_{t=2}^{3} e^{-[\sum_{\tau=1}^{t} u(c(\tau))]}}]$$

$$= \frac{u'(c(1))}{u'(c(3))} \cdot [\frac{1}{e^{-[u(c(2))+u(c(3))]}} + \frac{1}{e^{-u(c(3))}} + 1]$$

Along a sequence of three constant consumption levels, $\overline{c}$, the MRS is

$$MRS_{(c=\overline{c})} = \frac{u'(\overline{c})}{u'(\overline{c})} \cdot [1 + \frac{1}{e^{-[u(\overline{c})]}}]$$

and the time preference rate is

$$\rho_{1,3,(c=\overline{c})}(3) = \frac{1}{e^{-[u(\overline{c})]}}$$

5 Conclusions

This paper reviewed the Epstein and Hynes [3] model of endogenous time preference framed in continuous time and extended its formulation to discrete time to study intertemporal substitution between consumption at different times and derive closed form expressions for both the time preference rate and the utility discount rate.

References

[1] Gary S. Becker and Casey B. Mulligan. The endogenous determination of time preference. *The Quarterly Journal of Economics*, 112:729–758, 1997.

[2] Eugen V. Böhm-Bawerk. *The Positive Theory of Capital*, 1891 (accessed February 26, 2021). URL: https://oll-resources.s3.us-east-2.amazonaws.com/oll3/store/titles/283/0183_Bk.pdf.

[3] Larry Epstein and J. Allan Hynes. The rate of time preference and dynamic economic analysis. *Journal of Political Economy*, 91:611–35, 1983.

[4] Irving Fisher. *The Rate of Interest*, 1907 (accessed March 9, 2021). URL: https://socialsciences.mcmaster.ca/~econ/ugcm/3ll3/fisher/RateofInterest.pdf.

[5] Irving Fisher. *The Theory of Interest*, 1930 (accessed March 9, 2021). URL: http://files.libertyfund.org/files/1416/0219_Bk.pdf.

[6] Tjalling C. Koopmans. Stationary ordinal utility and impatience. *Econometrica*, 28:287–309, 1960.

[7] Hirofumi Uzawa. Time preference, the consumption function and optimum asset holdings. In J. N. Wolfe, editor, *Value, Capital and Growth: Papers in Honor of Sir John Hicks*, pages 485–504. Aldine Publishing Company, Chicago, 1968.