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A BAYESIAN ANALYSIS OF MULTIPLE CHANGES IN THE VARIANCE OF FIRST- ORDER AUTOREGRESSIVE TIME SERIES MODELS.

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ABSTRACT. The problem of a change in the mean of a sequence of random variables at an unknown time point has been addressed extensively in the literature. But, the problem of a change in the variance at an unknown time point has, however, been covered less widely. This paper analyses a sequence of first order autoregressive time series model in which the variance may have subjected to multiple changes at an unknown time points. Posterior distributions are found both for the unknown points of time at which the changes occurred and for the parameters of the model. A numerical example is illustrated.

Keywords: Time series model; Autoregressive model; Variance Change Posterior distribution.

1. INTRODUCTION

In recent times, inference problems associated with time series models with change point problems are increasingly met within the statistical analysis of many real life problems. In the study of change point the relationship between yield data and explanatory variables in growth models, dependence studies in chemical reactions, etc., it is very often noted that the relationship is of one type for a certain configuration of the values of the explanatory variables and of another type for a different configuration of the values of the explanatory variables. Such changes in the relationships are, some time sudden and some time gradual. In such circumstances, it is not possible to use the conventional theory of time series models which explicitly assumes a fixed rigid relationship throughout. Switching linear models are quite useful and provide better models for the data in such situations.

Consider a manufacturing industry producing a particular consumer product. The profit margin of the company may follow a particular pattern (per capita) until a period when a new technology is introduced or the workers are given specialized training in handling the machines. From that period onwards the profit margin (per capita) may show a new pattern. This is an example of a sudden structural change. In this example, the time point when a structural change takes place is clearly predictable. But, in many real life problems, this may not be possible and we may have to make inferences only on the basis of the data collected on the variables of interest.

In certain other situations, the structure of the models may begin to change either through the mean or variance of the errors at a particular period of time due to one or more reasons and the change may continue over a certain period of time at the end of which it might stabilize. Hsu(1977) examines the problem of testing whether there has been a change in the variance at an unknown time point using sampling theory, and applies to stock return data and also give a Bayesian treatment of a similar problem.

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Considerable work has been done in the recent past on the structural changes in regression models regarding the detection, estimation and inferences of switch points and parameters of switching linear regression models and sequences of normal, Poisson, Binomial and Gamma random variable. But very little work has been reported on switching time series models.

Switching first order autoregressive process with one change is defined as

$$X_t = \alpha_1 X_{t-1} + e_t ; \qquad t = 1,2, \ldots, t_1$$

$$X_t = \beta_1 X_{t-1} + e_t ; \qquad t = t_1 + 1,2, \ldots, n$$

where t_1 is the shift point, $t_1 = 1, 2, \ldots, n - 1$, α_1 and β_1 are the autoregressive parameter of before and after change respectively, e_t′s are identically independently distributed normal variables with mean zero and common variance σ^2 and other details can be found in Broemeling (1985).

The problem of variance changes in the AR(1) model is defined as

$$X_t \quad = \quad \beta X_{t-1} + e_t, \qquad t = 1, 2, \ldots, N$$

$$\text{with} \quad \operatorname{var}(e_t) = \begin{cases} \sigma_1^2 & ; \quad 1 \le t \le k \text{ and} \\ \sigma_2^2 & ; \quad k < t \le N \end{cases}$$

and other details can be found in Menzefricke (1981).

The organization of this paper is as follows. The Section I give a brief introduction about change point problems in time series models. Section II provides a brief review of change point problems. Section III provides the Bayesian inference to variance changes in the autoregressive time series models. Section IV provides the numerical study and Section V gives the brief summary and conclusion of the paper.

2. REVIEW OF LITERATURE

Problems involving switches have also been studied in the past under topics like change point problems, structural change, slippage and outliers, model selection, mixture distributions, cluster analysis, etc., in the statistical and econometric literature. A review of the published literature shows that some of the solutions to the slippage and outliers problems have been dealt with in the literature on structural change problems in the literature pertaining to mixture distributions and cluster analysis

The first Bayesian work in the area of switch point problem was by Chernoff and Zacks (1964) who introduced the Bayesian methodology to estimate the mean of the "Current variables" in multiple changes problem in a sequence of normal random variables by assuming the relationship between successive means to be

$$\mu_{i+1} = \mu_i + J_i z_i, \qquad i = 1, 2, \ldots, n - 1$$

where J_i is an indicator variable which assumes the value one if the change occurs between the time points i and i+1 and zero otherwise and Z_i is a random variable which represents the amount of change when a change takes place. They assumed a normal prior for the mean and the amount of change and a uniform prior for the change points. By assuming a quadratic loss function, the Bayesian estimator for the mean was derived.

Instead of the multiple changes model of Chernoff and Zacks (1964), Hinkley (1971) considered a model incorporating a single change in the mean of a sequence of normal random variables with constant known variance. A CUSUM test procedure was used to estimate the change point. Bayesian estimate of the change point with respect to a finite sequence of normal random variables with known σ was obtained by Broemeling (1972) who employed non informative prior for all the parameters. Later on, Broemeling(1974) studied the same problem employing conjugate priors for the parameters. He also used the Posterior odds ratio to test the hypothesis that the change point is a specified point. For the same model, assuming σ^2 to be unknown, Holbert(1982) derived the posterior distribution of the change point employing non informative prior for σ^2.

Smith (1975) considered a Bayesian approach to the problem of making inferences about the point of change in a sequence of random variables at which the underlying distribution changes and derived the Bayesian test procedure, the posterior to prior odds ratio, for testing the existence of abrupt changes in the Integrated Moving Average Model and Auto correlated Error Models employing non-informative prior for all the parameters

It is needless to point out that all the above works relate to abrupt changes in the means of a sequence of random variables. Hsu (1977) for the first time, studied the variance change problem with respect to a sequence of normal random variables and developed a locally most powerful test (LMPT) and a chi-square CUSUM test to test the hypothesis of no change against the alternative that there is a change in the variance at some point in the finite sequence of random variables. He concluded, on the basis of power comparisons, that the CUSUM chi-square test is a useful one for investigating the variance shift.

Salazar (1982) studied the gradual changes problem in the context of time series models, first and second order autoregressive process models, lagged variable model and auto correlated error models through the Bayesian approach employing non-informative prior for all the parameters except the precision parameter for which a gamma prior was employed. Bayesian estimators were obtained for all the parameters by utilizing numerical integration techniques to remove the nuisance parameter. Venkatesan and Arumugam (2005, 2007) derived the Bayesian estimates of the first order autoregressive model through the parameter changes and Venkatesan *et al*., (2009a, b) were studied the mean changes with respect to the autoregressive time series models with multiple changes and illustrated with numerical study.

3. MULTIPLE CHANGES IN THE VARIANCE OF FIRST- ORDER AR MODEL AND IT'S BAYESIAN ANALYSIS

This section is concerned with a study of the problem of changes in the variance of the time series model and the investigation of a Bayesian inference to the same.

Multiple changes in first order autoregressive model

First, let us consider the simple first order autoregressive time series model is given by

$$X_t = \beta X_{t-1} + e_t, \qquad t = 1, 2, \ldots, N \tag{3.1}$$

Consider a series $\{X_t\}$, t = 1, 2, …, N and let $\hat{n}_1, \hat{n}_2, ..., \hat{n}_m$ be preliminary estimates of the time points $n_1, n_2, .., n_m$ at which variance changes occur. Such estimates may be found by the method described by Wichern et al., (1976).

Define $\hat{n}_0 = n_0 = 1$ and $\hat{n}_{m+1} = N$ and

let var $(e_t) = \sigma_i^2$, $n_{i-1} < t \le n_i$; i = 1, 2, …, m + 1 and

$\sigma^2 = (\sigma^2_1, \sigma^2_2, \ldots, \sigma^2_{m+1})'$

Given m (number changes in the data) and $X = (X_1, X_2, \ldots, X_N)'$, and the problem of (i) estimating the autoregressive parameter (ii) estimating the variances and (iii) Bayesian estimates of the change points through the Bayesian methodology.

The likelihood function resulting from n observation is

$$P(X/\Theta) \propto \prod_{i=1}^{m+1} (\sigma_i^2)^{-(n_i - n_{i-1})/2} \,.\, \exp\{-S_i(\beta) / 2\sigma_i^2\}$$

Where $$S_i(\beta) = \sum_{t=n_{i-1}+1}^{n_i} (X_t - \beta X_{t-1})^2, \quad i = 1, 2, \ldots, m+1.$$

$$\Theta = (\beta, \sigma_1^2, \sigma_2^2 \ldots, \sigma^2_{m+1}, n_1 \ldots n_m) \tag{3.2}$$

To find the posterior distribution of Θ, one first have to specify the prior distribution for the parameters and is assumed as follows.

i) n_i, the change point, follows a discrete uniform distribution in its range.

ii) The autoregressive parameter ‘β’ follows a normal distribution with mean zero and unit variance.

iii) σ_i^2 follows inverted gamma distributions with parameters δ_i and γ_i.

Therefore, the joint prior distribution of the parameters of Θ is given by

$$P(\Theta) \propto \left[\prod_{i=1}^{m+1} (\sigma_i^2)^{-(\delta_i+1)} \,.\, \exp\left\{-\frac{\gamma_i}{\sigma_i^2}\right\}\right] e^{-1/2(\beta-a)^2} \tag{3.3}$$

The joint posterior distribution P(Θ / X) is proportional to the product of P(X/Θ) and P(Θ) are respectively given in equations (3.2) and (3.3) and after simplifications is given by

P(Θ/X)

$$\propto \quad e^{-\frac{1}{2}(\beta-a)^2}\left[\prod_{i=1}^{m+1}(\sigma_i^2)^{-\left(\frac{n_i-n_{i-1}}{2}+\delta_1+1\right)}\right] e^{-\sum_{i=1}^{m+1}\left(\frac{S_i(\beta)}{2\sigma_i^2}+\frac{\gamma_i}{\sigma_i^2}\right)} \tag{3.4}$$

where $S_i(\beta) = \sum_{t=n_{i-1}+1}^{n_i}(X_t-\beta X_{t-1})^2$

$$= \sum_{t=n_{i-1}+1}^{n_i}\left(X_t^2+\beta^2 X_{t-1}^2-2\beta X_t X_{t-1}\right)$$

$$= \Sigma X_t^2+\beta^2 \Sigma X_{t-1}^2-2\beta \, \Sigma X_t X_{t-1}$$

$$\frac{S_i(\beta)}{2}+\gamma_i=\frac{1}{2}\left[\beta^2 \sum_{t=n_{i-1}+1}^{n_i} X_{t-1}^2-2\beta \sum_{t=n_{i-1}+1}^{n_i} X_t X_{t-1}+\sum_{t=n_{i-1}+1}^{n_i} X_t^2+2\gamma_i\right]$$

$$= \frac{1}{2}\left[\beta^2 B_i-2\beta C_i+A_i\right]$$

where $B_i = \sum_{t=n_{i-1}+1}^{n_i} X_{t-1}^2$; $\quad C_i = \sum_{t=n_{i-1}+1}^{n_i} X_t X_{t-1}$;

$$A_i = \sum_{t=n_{i-1}+1}^{n_i} X_t^2+2\gamma_i$$

Therefore $S_i^*(\beta) = \frac{S_i(\beta)}{2}+\gamma_i=\frac{1}{2}\left[\beta^2 B_i-2\beta C_i+A_i\right]$

Hence, P(Θ/X) $\propto$ $$e^{-\frac{1}{2}(\beta-a)^2}\left[\prod_{i=1}^{m+1}(\sigma_i^2)^{-\left(\frac{n_i-n_{i-1}}{2}+\delta_i+1\right)}\right]e^{-\sum_{i=1}^{m+1}S_i^*(\beta)/2\sigma_i^2} \quad (3.5)$$

where $S_i^*(\beta)=\beta^2 B_i-2\beta C_i+A_i$; i = 1, 2, …, m+1, and Θ = (n_1, n_2, …, n_m, β, $\sigma_1^2,\ldots,\sigma_{m+1}^2$)

Integrating out σ_i^2 out of this joint posterior distribution, we obtain the joint posterior distribution of β_1, n_1, … , n_m, $\sigma_1^2,\sigma_{i-1}^2,\ \sigma_{i+1}^2,\ldots,\sigma_{m+1}^2$ and is obtained as

$$\propto e^{-\frac{1}{2}(\beta-a)}\prod_{\substack{j=1\\ i\neq j}}^{m+1}\left[(\sigma_j^2)^{-\left(\frac{n_j-n_{j-1}}{2}+\delta_j+1\right)}\right]\left[e^{-\sum_{\substack{j=1\\ j\neq i}}^{m+1}S_j^*(\beta)/2\sigma_j}\right]\frac{\overline{(n_i}-n_{i-1}+\delta_i)}{\left[S_i^*(\beta)/2\right]^{\frac{n_i-n_{i-1}}{2}+\delta_i}} \quad (3.6)$$

The above joint probability density function, (3.6) is a very complicated expression to simplify, and similarly by integrate out each σ_j^2, one may get

$$P(\beta, n_1, n_2, \ldots, n_m) \propto e^{-\frac{1}{2}(\beta-a)^2}\prod_{i=1}^{m+1}\frac{\overline{(n_i}-n_{i-1}+\delta_i)}{\left[S_i^*(\beta)/2\right]^{\frac{n_i-n_{i-1}}{2}+\delta_i}} \quad (3.7)$$

The above expression is very complicated and is analytically not tractable. One-way of solving the problem is to find the marginal posterior distribution using MCMC technique.

4. NUMERICAL STUDY

In order to illustrate the solutions of the change point problems described in section 3 a computer study was carried out. The main aim of the numerical study is to illustrate the evaluation of the estimates of the parameters on the basis of the methodology developed.

In this paper MCMC technique is used to compute marginal posterior distributions and Bayes estimates of the parameters. Markov Chain Monte Carlo (MCMC) is a powerful technique for performing integration by simulation. In recent years MCMC has revolutionized the application of Bayesian statistics. Many high dimensional complex models, which were formally intractable, can now be handled routinely. Bayesian calculations not analytically tractable can be performed using MCMC once a likelihood and prior are given. MCMC simulation algorithm in its basic form is quite simple and becoming standards in much Bayesian application. One of the basic goals of general Bayesian framework is to compute expectations with respect to a high dimensional probability distribution.

The two main algorithms used in MCMC applications are (i) Gibbs Sampler and (ii) Metropolis algorithms. All other algorithms are generalization of Metropolis. The simplest MCMC algorithm is a Gibbs sampler. Gibbs sampler iteratively samples each variable conditional on the most recent value of all other variables.

The analytical solution is not available for the posterior density function since the joint posterior distribution is a complicated function of the parameters. Therefore, one may have to resort to numerical integration or MCMC or Gibbs sampling technique to determine the estimates.

Variance changes in time series data is illustrated by following. Box and Jenkins (1970) fitted the ARIMA (0, 1, 1) model to a series of 369 IBM stock prices. The same data is fitted through the methodology suggested in section 3 by the first difference of the same series (logarithm). Further due to certain practical limitations in computing, attention was focused on the switching first order autoregressive process. The point estimates of the parameters were evaluated numerically by taking the posterior mean as the estimate. MCMC technique is used to evaluate the marginal posterior distribution and hence the Bayes estimates.We propose an AR(1) model with two innovations variance changes and is given in the following results:

$$\hat{\alpha} = 0.13, \qquad 10^4 \operatorname{var}(e_t) = \begin{cases} 0.99 \; ; & t < 179 \\ 0.60 \; ; & 180 \le t \le 235 \\ 0.72 \; ; & t \ge 235 \end{cases}$$

5. RESULTS AND CONCLUSIONS

This paper is the output of an investigation regarding the building up of suitable models to represent the structural changes in time series models. Bayesian methodology has been used in making inferences about the parameters of the model. A Bayesian solution of the change point problems through the variance change in autoregressive models is discussed and a numerical study has been illustrated through the procedure developed and also to examine the quality of the results obtained. Generally, the estimates are found to be close to the true values using which IBM stock price data.

The estimates are quite close to the true values when the magnitude of the switch is large relative to the variance.

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