# Monitoring Industrial Process using a Robust Modified Mean Chart

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## Abstract

Shewhart control chart is the most popular and widely used Statistical process Control tool to monitor process. It is developed under the assumption of independent and normally distributed process. In order to control process mean and standard deviation, robust estimator of these parameters can be better alternatives as charts based on that are more resistant to moderate changes in process distribution. Modified Maximum Likelihood Estimator (MMLE) for mean and standard deviation is a pair of statistics with good robust properties. Authors introduced these measures to control charting process and investigate the advantages of using it. A modification to mean based on MMLE and its standard deviation are introduced to improve industrial process performance. Using Monte Carlo simulation method, performance of this chart is compared with classical control chart. Performance is also studied based on the Average Run Length.

*Keywords*: control chart, mean, standard deviation, modified maximum likelihood estimator, average run length.

## 1. Introduction

Specifying the control limits is the most important step in designing a control chart. When the limits are narrow, the risk of a point falling beyond the limits increases, and hence increase the false indication that the process is out of control. If the limits are wider, the risk increases the points falling within the limits, falsely indicating that the process is in control. The presence of outliers tends to reduce the sensitivity of control-charting procedures because the control limits become stretched so that the detection of the outliers themselves becomes more difficult. Width of the control limits are also depends on the statistic used to construct the limits. Hence the usage of some resistant statistics in constructing control limits can improve the performance of charts in detection of assignable causes of variation as well as variation due to the presence of outliers. Robustness depends not only on resistance of the statistical tools to outliers but also on the purpose of the analysis.

The most common Shewhart type control chart to control a quality characteristic at its target value in an industrial process, is a mean chart. In practice, the process parameters $\mu$ and $\sigma$ are usually unknown. Therefore, they must be estimated from preliminary subgroups taken when the process is assumed to be in control and limits are constructed based on it

in Phase I analysis. The resulting estimates and limits are used to monitor the location of the process in Phase II. The $\bar{x}$ chart is developed under the assumption that the data collected from the process is independently and identically distributed normal distribution. The quality of subgroup average is easily affected by the presence of extreme values. There are many resistant statistics available in literature and some of them are experimented to control industrial process. Langenberg and Iglewicz (1986) proposed control limits determined by the trimmed mean of the subgroup means and the trimmed mean of the ranges. Rocke (1989, 1992) proposed trimmed Mean and Inter Quartile Range chart and Median and Range chart to control process mean. Schoonhoven *et al* (2011) have used Tatum estimator and trimean value based on quantiles. Wu *et al* (2002) considered three alternative statistics for the sample standard deviation, namely the median of the absolute deviation from the median (MDM), the average absolute deviation from the median (ADM) and the median of the average absolute deviation (MAD), and investigated their effect on $\bar{x}$ control chart performance. Omar Ahmed Abu-Shawiesh (2009) used the sample median, MD, to estimate the process mean, $\mu$, and MAD for dispersion $\sigma$. Jones-Farmer *et al* (2019) proposed a rank-based Phase I location control chart. Figueiredo and Gomes (2009) have investigated Bootstrap based median and range and total median and total range statistics to control process mean. Maddahi *et al* (2011) presented a robust mean control chart based on M-estimators in presence of outliers. Adekeye (2012), Adekeye and Azubuike (2012) modified sample mean chart with standard deviation estimated using MAD and showed that the chart is good even for non-normal process. Schoonhoven *et al* (2013) used trimean (TM) and Adaptive Trimmed Standard deviation (ATS) to control process. This paper gives a modification to $\bar{x}$ chart using MMLE to make the control limits robust. Nazir *et al* (2014a,b) gave step-by-step procedure to develop a robust Shewhart location and dispersion control charts. Lagos-Álvarez *et al* (2014) have used MMLE to estimate parameters of generalization of the symmetrical linear regression models. Sindhumol and Srinivasan (2015) introduced dispersion chart based on MMLE. Sindhumol *et al* (2016a,b) developed control charts for location and dispersion based on trimmed mean and Modified Trimmed Standard Deviation (MTSD).

This paper presents robust location charts based on Modified Maximum Likelihood Estimator. This location measure has a problem that the tails of distribution can dominate its value, especially in presence of outliers. This problem is overcome by a modification to the estimator giving less weight to the values in the tails and hence more attention to values near center. This robust estimator is used to develop control limits and its performance is also compared with classical mean chart.

## 2. Modified maximum likelihood estimators (MMLE)

Consider the normal distribution $N(\mu, \sigma)$ and let $\underline{x}^T = (x_1, x_2, ..., x_n)$ be a random sample of size $n$ taken on it. If $z = (x_{(i)} - \mu)/\sigma$, $f(z)$ is the standard normal distribution based on symmetric censored sample ($x_{(i)}$ for $i = r + 1, ..., n - r$), with the likelihood function given as

$$L = \frac{n!}{2r!}\sigma^{-(n-2r)}\left[\prod_{i=r+1}^{n-r} f(z_{(i)})\right]\left[F(z_{(r+1)})\right]^r\left[1 - F(z_{(n-r)})\right]^r. \tag{1}$$

Thus, the maximum likelihood equations are

$$\frac{\partial log L}{\partial \mu} = \frac{n}{\sigma}\left\{\frac{1}{n}\sum_{i=r+1}^{n-r} z_{(i)} - q\left[g_1(z_{(r+1)}) - g_2(z_{(n-r)})\right]\right\} = 0 \tag{2}$$

$$\frac{\partial log L}{\partial \sigma} = \frac{n}{\sigma}\left\{-(1-2q) + \frac{1}{n}\sum_{i=r+1}^{n-r} z_{(i)}^2 - q\left[z_{(r+1)}g_1(z_{(r+1)}) - z_{(n-r)}g_2(z_{(n-r)})\right]\right\} = 0 \tag{3}$$

where $q = r/n$; $g_1(z) = f(z)/F(z)$ and $g_2(z) = f(z)/[1 - F(z)]$. The equations 2 and 3 do not have explicit solutions and Cohen (1961) has developed a suitable iterative method for

solving these equations. Tiku (1967) considered $g(z) = f(z)/[1 - \text{F}(z)] = \alpha + \beta z$ where $\beta = (g(b) - g(a))/(b - a)$ and $\alpha = g(z) - \alpha\beta$, as $g(z)$ over an interval $a \leq z \leq b$ of finite length lie close to the line. Tiku *et al* (1986) introduced Modified Maximum Likelihood Estimators (MMLE) of mean and standard deviation of normal distribution as

$$
\begin{aligned}
\hat{\mu}_{\text{MMLE}} = \bar{x}_{\text{MMLE}} &= \frac{\sum_{i=r+1}^{n-r} x_{(i)} + r\beta(x_{(r+1)} + x_{(n-r)})}{m}, \text{ and} \\
\hat{\sigma}_{\text{MMLE}} = s_{\text{MMLE}} &= \frac{B + (B^2 + 4AC)^{1/2}}{2[A(A-1)]^{1/2}}.
\end{aligned}
\tag{4}
$$

where $n$ =sample size, $m = n - 2r + 2r\beta$, $A = n - 2r$, $B = r\alpha(x_{(n-r)} - x_{(r+1)})$ and $C = \sum_{i=r+1}^{n-r}(x_{(i)} - \bar{x}_{\text{MMLE}})^2 + r\beta[(x_{(r+1)} - \bar{x}_{\text{MMLE}})^2 + (x_{(n-r)} - \bar{x}_{\text{MMLE}})^2]$.

He has derived values of $\alpha$ and $\beta$ for various $n$ and $r$ and identified the MMLE estimators as robust estimators for mean $\mu$ and standard deviation $\sigma$. However, when symmetric censoring $r = 0$, these estimates are reduced to MLE. He studied its robust nature and showed that MMLE (30% censoring) is preferable to sample mean and standard deviation for symmetric distributions with infinite mean and variance and MMLE (10% censoring) is preferable to extreme short tailed symmetric distributions. He introduced a link between censoring and robustness also. For Type II censored normal samples, the MMLEs the pair ($\hat{\mu}$ and $\hat{\sigma}$) are all most equally efficient as the MLEs and BLUEs. More than that, the MMLEs are jointly more efficient than trimmed estimators, Huber's and Humpel's estimators.

## 3. Robust location control charts

If sample mean and standard deviation are used to estimate process means $\mu$ and process dispersion $\sigma$, based on $k$ preliminary samples each of size $n$, control limits of chart namely Center Line (CL), Upper Control Limit (UCL) and Lower Control Limits (LCL) are obtained as follows by calculating average of $m$ sample means and sample standard deviations.

$$
\text{CL} = \bar{\bar{x}}; \quad \text{UCL} = \bar{\bar{x}} + 3\frac{\bar{s}}{c_4\sqrt{n}} = \bar{\bar{x}} + A_3\bar{s}; \quad \text{LCL} = \bar{\bar{x}} - 3\frac{\bar{s}}{c_4\sqrt{n}} = \bar{\bar{x}} - A_3\bar{s}.
\tag{5}
$$

If sample MMLE is used to estimate $\mu$ and standard deviation $s$ is used to estimate process dispersion $\sigma$, control limits are

$$
\text{CL} = \bar{\bar{x}}_{\text{MMLE}}; \quad \text{UCL} = \bar{\bar{x}}_{\text{MMLE}} + A_3\bar{s}; \quad \text{LCL} = \bar{\bar{x}}_{\text{MMLE}} - A_3\bar{s}.
\tag{6}
$$

If sample MMLE and standard deviation of MMLE are used to estimate process $\mu$ and dispersion $\sigma$, control limits are

$$
\begin{aligned}
\text{CL} &= \bar{\bar{x}}_{\text{MMLE}}; \\
\text{UCL} &= \bar{\bar{x}}_{\text{MMLE}} + 3\frac{\bar{s}_{\text{MMLE}}}{\sqrt{m}} = \bar{\bar{x}}_{\text{MMLE}} + A_4\bar{s}_{\text{MMLE}}; \\
\text{LCL} &= \bar{\bar{x}}_{\text{MMLE}} - 3\frac{\bar{s}_{\text{MMLE}}}{\sqrt{m}} = \bar{\bar{x}}_{\text{MMLE}} + A_4\bar{s}_{\text{MMLE}}.
\end{aligned}
\tag{7}
$$

Values of constant $A_4$ are calculated and are given in Table 1 for various sample sizes $n$, $r$, $m$, $\alpha$ and $\beta$. This table is presented as an extension of Table B.1 of Tiku *et al* (1986), where values of $\alpha$ and $\beta$ for various $n$ and $r$, were given, for the convenience of the user.

The comparison of control limits for performance evaluation is carried out based on twenty $k$ simulated samples of size ten ($n$) each from $N(0,1)$ with $r = 0, 1$ and 2. Limits are constructed repeatedly for such simulated data and average width of control limits is given in Table 2.

Table 1: Control chart parameters for MMLE-chart (Part 1)

| $n$ | $r$ | $\alpha$ | $\beta$ | $m$ | $A_4$ | $n$ | $r$ | $\alpha$ | $\beta$ | $m$ | $A_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 1 | 0.754 | 0.8108 | 4.6216 | 1.395485 | 21 | 1 | 0.6104 | 0.8915 | 20.783 | 0.658062 |
| 6 | 1 | 0.7303 | 0.8273 | 5.6546 | 1.261596 | 21 | 2 | 0.6826 | 0.8427 | 20.3708 | 0.664687 |
| 7 | 1 | 0.7118 | 0.8391 | 6.6782 | 1.160891 | 21 | 3 | 0.7166 | 0.8148 | 19.8888 | 0.672693 |
| 7 | 2 | 0.7867 | 0.7508 | 6.072 | 1.217462 | 21 | 4 | 0.7411 | 0.7903 | 19.3224 | 0.682481 |
| 8 | 1 | 0.6968 | 0.8481 | 7.6962 | 1.081392 | 21 | 5 | 0.76 | 0.7672 | 18.672 | 0.694266 |
| 8 | 2 | 0.7719 | 0.768 | 7.072 | 1.128107 | 21 | 6 | 0.7751 | 0.7445 | 17.934 | 0.708407 |
| 9 | 1 | 0.6844 | 0.8552 | 7.7104 | 1.080396 | 22 | 1 | 0.607 | 0.893 | 21.786 | 0.642736 |
| 9 | 2 | 0.7594 | 0.7811 | 8.1244 | 1.052508 | 22 | 2 | 0.679 | 0.845 | 21.38 | 0.64881 |
| 9 | 3 | 0.7959 | 0.7242 | 7.3452 | 1.106928 | 22 | 3 | 0.7127 | 0.818 | 20.908 | 0.656092 |
| 10 | 1 | 0.6737 | 0.861 | 9.722 | 0.962151 | 22 | 4 | 0.737 | 0.7944 | 20.3552 | 0.664942 |
| 10 | 2 | 0.7485 | 0.7915 | 9.166 | 0.990903 | 22 | 5 | 0.7559 | 0.7722 | 19.722 | 0.675532 |
| 10 | 3 | 0.7855 | 0.7405 | 8.443 | 1.032459 | 22 | 6 | 0.7711 | 0.7506 | 19.0072 | 0.688117 |
| 11 | 1 | 0.6645 | 0.8658 | 10.7316 | 0.915775 | 22 | 7 | 0.7832 | 0.7289 | 18.2046 | 0.703122 |
| 11 | 2 | 0.7391 | 0.8001 | 10.2004 | 0.939318 | 23 | 1 | 0.6038 | 0.8944 | 22.7888 | 0.628435 |
| 11 | 3 | 0.7762 | 0.7535 | 9.521 | 0.972255 | 23 | 2 | 0.6756 | 0.8472 | 22.3888 | 0.634024 |
| 12 | 1 | 0.6564 | 0.87 | 11.74 | 0.875563 | 23 | 3 | 0.709 | 0.8209 | 21.9254 | 0.640689 |
| 12 | 2 | 0.7307 | 0.8072 | 11.2288 | 0.895271 | 23 | 4 | 0.7331 | 0.7981 | 21.3848 | 0.648737 |
| 12 | 3 | 0.7677 | 0.7643 | 10.5858 | 0.92206 | 23 | 5 | 0.752 | 0.7768 | 20.768 | 0.6583 |
| 12 | 4 | 0.7919 | 0.723 | 9.784 | 0.959098 | 23 | 6 | 0.7672 | 0.7561 | 20.0732 | 0.669596 |
| 13 | 1 | 0.6492 | 0.8736 | 12.7472 | 0.84026 | 23 | 7 | 0.7796 | 0.7354 | 19.2956 | 0.682955 |
| 13 | 2 | 0.7232 | 0.8133 | 12.2532 | 0.857031 | 24 | 1 | 0.6008 | 0.8957 | 23.7914 | 0.615051 |
| 13 | 3 | 0.7599 | 0.7733 | 11.6398 | 0.879323 | 24 | 2 | 0.6724 | 0.8492 | 23.3968 | 0.620216 |
| 13 | 4 | 0.7847 | 0.7354 | 10.8832 | 0.909375 | 24 | 3 | 0.7055 | 0.8236 | 22.9416 | 0.626339 |
| 14 | 1 | 0.6428 | 0.8767 | 13.7534 | 0.80894 | 24 | 4 | 0.7295 | 0.8016 | 22.4128 | 0.633685 |
| 14 | 2 | 0.7165 | 0.8186 | 13.2744 | 0.823406 | 24 | 5 | 0.7483 | 0.781 | 21.81 | 0.642382 |
| 14 | 3 | 0.7529 | 0.7809 | 12.6854 | 0.842305 | 24 | 6 | 0.7636 | 0.7611 | 21.1332 | 0.652587 |
| 14 | 4 | 0.7779 | 0.7459 | 11.9672 | 0.867211 | 24 | 7 | 0.776 | 0.7413 | 20.3782 | 0.664566 |
| 15 | 1 | 0.6369 | 0.8795 | 14.759 | 0.780895 | 25 | 1 | 0.5979 | 0.8968 | 24.7936 | 0.602492 |
| 15 | 2 | 0.7103 | 0.8233 | 14.2932 | 0.793518 | 25 | 2 | 0.6694 | 0.851 | 24.404 | 0.607282 |
| 15 | 3 | 0.7464 | 0.7876 | 13.7256 | 0.809759 | 25 | 3 | 0.7023 | 0.8261 | 23.9566 | 0.612927 |
| 15 | 4 | 0.7715 | 0.7518 | 13.0144 | 0.83159 | 25 | 4 | 0.726 | 0.8048 | 23.4384 | 0.619665 |
| 15 | 5 | 0.7896 | 0.7223 | 12.223 | 0.858089 | 25 | 5 | 0.7448 | 0.7849 | 22.849 | 0.627607 |
| 16 | 1 | 0.6316 | 0.882 | 15.764 | 0.755593 | 25 | 6 | 0.76 | 0.7657 | 22.1884 | 0.636881 |
| 16 | 2 | 0.7048 | 0.8274 | 15.3096 | 0.766724 | 25 | 7 | 0.7726 | 0.7468 | 21.4552 | 0.647672 |
| 16 | 3 | 0.7405 | 0.7935 | 14.761 | 0.780842 | 25 | 8 | 0.7829 | 0.7278 | 20.6448 | 0.660261 |
| 16 | 4 | 0.7656 | 0.7626 | 14.1008 | 0.798913 | 26 | 1 | 0.5952 | 0.8968 | 25.7936 | 0.590698 |
| 16 | 5 | 0.784 | 0.7324 | 13.324 | 0.821872 | 26 | 2 | 0.6666 | 0.851 | 25.404 | 0.59521 |
| 17 | 1 | 0.6267 | 0.8843 | 16.7686 | 0.73261 | 26 | 3 | 0.6991 | 0.8261 | 24.9566 | 0.600521 |
| 17 | 2 | 0.6996 | 0.8311 | 16.3244 | 0.742511 | 26 | 4 | 0.7227 | 0.8048 | 24.4384 | 0.606855 |
| 17 | 3 | 0.735 | 0.7987 | 15.7922 | 0.754918 | 26 | 5 | 0.7414 | 0.7849 | 23.849 | 0.614308 |
| 17 | 4 | 0.76 | 0.7695 | 15.156 | 0.7706 | 26 | 6 | 0.7566 | 0.7657 | 23.1884 | 0.622997 |
| 17 | 5 | 0.7788 | 0.7411 | 14.411 | 0.790268 | 26 | 7 | 0.7693 | 0.7468 | 22.4552 | 0.633086 |
| 18 | 1 | 0.6221 | 0.8863 | 17.7726 | 0.711616 | 26 | 8 | 0.7798 | 0.7278 | 21.6448 | 0.644829 |
| 18 | 2 | 0.6949 | 0.8344 | 17.3376 | 0.720488 | 27 | 1 | 0.5926 | 0.899 | 26.798 | 0.579522 |
| 18 | 3 | 0.7299 | 0.8033 | 16.8198 | 0.731494 | 27 | 2 | 0.6638 | 0.8544 | 26.4176 | 0.58368 |
| 18 | 4 | 0.7548 | 0.7755 | 16.204 | 0.745264 | 27 | 3 | 0.6962 | 0.8307 | 25.9842 | 0.588527 |
| 18 | 5 | 0.7737 | 0.7488 | 15.488 | 0.762296 | 27 | 4 | 0.7196 | 0.8105 | 25.484 | 0.594275 |
| 19 | 1 | 0.618 | 0.8882 | 18.7764 | 0.692333 | 27 | 5 | 0.7381 | 0.7918 | 24.918 | 0.600986 |
| 19 | 2 | 0.6905 | 0.8374 | 18.3496 | 0.700338 | 27 | 6 | 0.7534 | 0.774 | 24.288 | 0.608731 |
| 19 | 3 | 0.7252 | 0.8075 | 17.845 | 0.710171 | 27 | 7 | 0.7661 | 0.7564 | 23.5896 | 0.617676 |
| 19 | 4 | 0.75 | 0.781 | 17.248 | 0.722357 | 27 | 8 | 0.7767 | 0.7389 | 22.8224 | 0.627972 |
| 19 | 5 | 0.7689 | 0.7556 | 16.556 | 0.737299 | 28 | 1 | 0.5902 | 0.9 | 27.8 | 0.568982 |
| 19 | 6 | 0.7836 | 0.7302 | 15.7624 | 0.755632 | 28 | 2 | 0.6613 | 0.856 | 27.424 | 0.57287 |
| 20 | 1 | 0.6141 | 0.8899 | 19.7798 | 0.674544 | 28 | 3 | 0.6933 | 0.8328 | 26.9968 | 0.577384 |
| 20 | 2 | 0.6864 | 0.8401 | 19.3604 | 0.681811 | 28 | 4 | 0.7166 | 0.8131 | 26.5048 | 0.582719 |
| 20 | 3 | 0.7208 | 0.8113 | 18.8678 | 0.690654 | 28 | 5 | 0.735 | 0.795 | 25.95 | 0.588915 |
| 20 | 4 | 0.7454 | 0.7859 | 18.2872 | 0.701532 | 28 | 6 | 0.7502 | 0.7776 | 25.3312 | 0.596065 |
| 20 | 5 | 0.7644 | 0.7617 | 17.617 | 0.714752 | 28 | 7 | 0.763 | 0.7607 | 24.6498 | 0.604247 |
| 20 | 6 | 0.7793 | 0.7378 | 16.8536 | 0.73076 | 28 | 8 | 0.7737 | 0.7438 | 23.9008 | 0.613642 |

Control chart parameters for MMLE-chart (Part 2)

| $n$ | $r$ | $\alpha$ | $\beta$ | $m$ | $A_4$ | $n$ | $r$ | $\alpha$ | $\beta$ | $m$ | $A_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 29 | 1 | 0.5878 | 0.901 | 28.802 | 0.558998 | 30 | 1 | 0.5856 | 0.9019 | 29.8038 | 0.549522 |
| 29 | 2 | 0.6588 | 0.8574 | 28.4296 | 0.562647 | 30 | 2 | 0.6564 | 0.8588 | 29.4352 | 0.552952 |
| 29 | 3 | 0.6906 | 0.8347 | 28.0082 | 0.566864 | 30 | 3 | 0.688 | 0.8365 | 29.019 | 0.556904 |
| 29 | 4 | 0.7137 | 0.8155 | 27.524 | 0.571828 | 30 | 4 | 0.711 | 0.8178 | 28.5424 | 0.561534 |
| 29 | 5 | 0.7321 | 0.7979 | 26.979 | 0.577575 | 30 | 5 | 0.7292 | 0.8006 | 28.006 | 0.566886 |
| 29 | 6 | 0.7472 | 0.7811 | 26.3732 | 0.584171 | 30 | 6 | 0.7443 | 0.7843 | 27.4116 | 0.572999 |
| 29 | 7 | 0.76 | 0.7647 | 25.7058 | 0.591706 | 30 | 7 | 0.7571 | 0.7684 | 26.7576 | 0.57996 |
| 29 | 8 | 0.7708 | 0.7484 | 24.9744 | 0.600307 | 30 | 8 | 0.768 | 0.7527 | 26.0432 | 0.58786 |
| 29 | 9 | 0.78 | 0.7321 | 24.1778 | 0.610117 | 30 | 9 | 0.7772 | 0.737 | 25.266 | 0.596833 |

Average width of the control limits based on MMLE is larger than that of classical control charts and the width increases further for $r = 2$. Though the pair of mean and s.d based on MMLE is robust, it cannot be directly applied to an industrial process.

Table 2: Average width of the charts.

| **Charts** | $\bar{x} - s$ | $\bar{x} - s_{\text{MMLE}}$ | | $\bar{x}_{\text{MMLE}} - s$ | | $\bar{x}_{\text{MMLE}} - s_{\text{MMLE}}$ | |
|---|---|---|---|---|---|---|---|
| | $r = 0$ | $r = 1$ | $r = 2$ | $r = 1$ | $r = 2$ | $r = 1$ | $r = 2$ |
| UCL | 0.937939 | 0.953736 | 0.963239 | 0.926557 | 0.921751 | 0.942352 | 0.947051 |
| LCL | -0.97035 | -0.98615 | -0.99565 | -0.98173 | -0.98654 | -0.99753 | -1.01184 |
| Width | 1.90829 | 1.939882 | 1.958888 | 1.908291 | 1.908288 | 1.939883 | 1.958888 |

## 3.1. Modifications of MMLE for control chart

When MMLE is introduced to industrial applications, it is noticed that as censoring percentage increases, width of the chart also increases and presence of an outlier makes limits even wider than non-robust mean chart. This is basically due to the large weightage given to the extreme observations, in order to overcome the effect of censoring. In order to reduce this effect, authors considered reduced weightage to extreme observations and equal weightage to other sample values. This reduction in weights of extreme observations positively affect sample dispersion and hence provide an overall improvement in width of control limits as well as the performance of chart. This improvement in limits of the mean chart, especially in phase I analysis, when some extreme observations are present and effect of this change is also reflected in phase II analysis too. Thus the modified MMLE are given by;

$$\hat{\mu} = \bar{x}^* = \frac{\sum_{i=r+1}^{n-r} x_{(i)} - r\beta(x_{(r+1)} + x_{(n-r)})}{m}, \text{ and}$$
$$\hat{\sigma} = s^* = \frac{B + (B^2 + 4AC)^{1/2}}{2[A(A-1)]^{1/2}}. \tag{8}$$

where $n$ = sample size, $m = n - 2r + 2r\beta$, $A = n - 2r$, $B = r\alpha(x_{n-r} - x_{r+1})$ and $C = \sum_{i=r+1}^{n-r}(x_{(i)} - \bar{x}^*)^2 - r\beta[(x_{(r+1)} - \bar{x}^*)^2 + (x_{(n-r)} - \bar{x}^*)^2]$.

The performance evaluation for the modified MMLE is carried out through a simulation study conducted by generating samples of sizes $n = 10$ and 20 from $N(0, 1)$. Sample mean $\bar{x}$ and robust $\bar{x}^*$ were calculated for 10,000 runs simulated for each sample using SAS software. Variance of $\bar{x}$ and $\bar{x}^*$ were calculated across the runs and Relative Efficiency (RE) of $\bar{x}^*$ with $\bar{x}$ were calculated using Mean Square Error (MSE). The results of the simulation study as presented in Table 3 showed that efficiency will improve if sample size and censoring percentage are increased.

Table 3: Relative efficiency of the modified robust estimator $\overline{x}^*$.

| Sample size $n$ | RE (10%) | Simulation Error | RE (20%) | Simulation Error |
|:---:|:---:|:---:|:---:|:---:|
| 10 | 2.15675 | 0.01445 | 0.89853 | 0.00598 |
| 20 | 1.41354 | 0.00709 | 2.046963 | 0.01608 |

If $\overline{x}^*$ is used to estimate $\mu$ and $s$ is used to estimate process dispersion $\sigma$, control limits are

$$\text{CL} = \overline{\overline{x}}^*; \quad \text{UCL} = \overline{\overline{x}}^* + A_3\overline{s}; \quad \text{LCL} = \overline{\overline{x}}^* - A_3\overline{s}. \tag{9}$$

If process standard is also estimated using $\overline{s}^*$, limits are

$$\text{CL} = \overline{\overline{x}}^*; \quad \text{UCL} = \overline{\overline{x}}^* + A_4\overline{s}^*; \quad \text{LCL} = \overline{\overline{x}}^* - A_4\overline{s}^*. \tag{10}$$

Table 4 gives comparison of charts based on its width using data simulated from $N(0,1)$ to study the effect without contamination. Charts based on MMLE and its standard deviation as well as modified mean and its standard deviation for censoring levels $r = 1$ and $r = 2$, for subgroup size $n = 10$ and number of subgroup $k = 20$ are compared with classical mean and its standard deviation chart. When compared with charts $\overline{x} - s$ and $\overline{x}_{\text{MMLE}} - s_{\text{MMLE}}$, $\overline{x}^* - s^*$ chart has a smaller width which helps an earlier detection of assignable causes. Width of the chart is also depended on what measure is used to estimate process dispersion.

Table 4: Average width of control charts for $n = 10$, $k = 20$.

| Charts | $\overline{x} - s$ | $\overline{x} - s^*$ | $\overline{x} - s_{\text{MMLE}}$ | $\overline{x}_{\text{MMLE}} - s$ | $\overline{x}_{\text{MMLE}} - s_{\text{MMLE}}$ | $\overline{x}^* - s$ | $\overline{x}^* - s^*$ |
|:---:|:---:|:---:|:---:|:---:|:---:|:---:|:---:|
| UCL | 0.91706 | 0.57536 | 0.95252 | 0.89474 | 0.93019 | 0.89609 | 0.55439 |
| LCL | -0.93416 | -0.59247 | -0.96962 | -0.95648 | -0.99194 | -0.95513 | -0.61344 |
| ($r = 1$) | **1.85121** | **1.16783** | **1.92214** | **1.85121** | **1.92214** | **1.85121** | **1.16783** |
| UCL | 0.91706 | 0.36633 | 1.01029 | 0.87055 | 0.96379 | 0.87055 | 0.31983 |
| LCL | -0.93416 | -0.38344 | -1.02739 | -0.98066 | -1.07389 | -0.98066 | -0.42994 |
| ($r = 2$) | **1.851214** | **0.74977** | **2.03768** | **1.85121** | **2.03768** | **1.85121** | **0.74977** |

Further, in order to study the robust nature of $\overline{x}^* - s^*$ chart control limits, it is compared with limits of $\overline{x} - s$ and $\overline{x}_{MMLE} - s_{MMLE}$ charts, in presence of contaminated data. Figure 1 shows that chart based on MMLE has the largest width and it fails to detect variation in the data due to assignable causes. Though LCL of $\overline{x}^* - s^*$ and $\overline{x} - s$ charts coincide and detect sample 3 and 4, UCL of $\overline{x}^* - s^*$ chart detects sample 18 and 20 also. Even though Tiku *et al* (1986) proved MMLE and its s.d are a pair of robust measures, it shows poor performance in control charting procedure than classical mean chart. The modified measures give robust control limits.

Performance of $\overline{x}^* - s^*$ charts in the presence of outliers is studied based on simulated data by taking 20 subgroups each of sizes $n = 10$ and $n = 20$ and outliers are fixed based on mean slippage and variance slippage methods. Simulated data from $N(0,1)$ is used for calculating control limits and two levels of censoring, 10% and 20% are considered. Outliers for mean slippage and variance slippage are considered from $N(1,1)$, $N(2,1)$, $N(0,2)$ and $N(1,2)$. Four control charts namely $\overline{x} - s$, $\overline{x} - s^*$, $\overline{x}^* - s$ and $\overline{x}^* - s^*$ are considered. The study showed that when outliers are presented from distributions having larger mean or standard deviation, almost all charts are detecting. Even when small shifts occur, this modified robust mean chart helps an early detection.

Figure 2 shows a comparison of these charts with classical mean charts. Among the four charts, three of them are based on modified mean or modified s.d. Charts based on modified mean are small in width and are overlapped while charts based on sample mean are wider. Due to the small width, robust limits help an early detection of shifts.
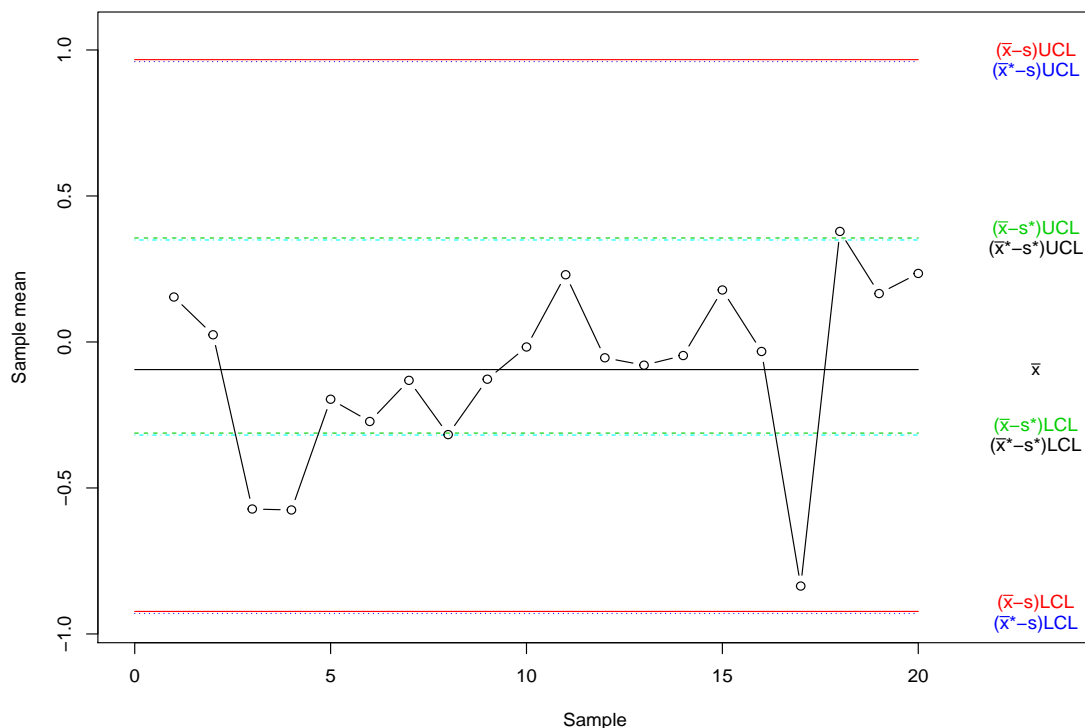
Figure 1: Comparison of contaminated control charts for $n = 10$, $k = 20$ and $r = 1$.

Figure 3 and 4 made comparison of charts based on outlier detection, where outliers are fixed using location and dispersion slippage methods respectively. Charts are developed based on 20 subgroups each of size 10 with 20% censoring. All the three charts based on modified mean and s.d are compared with classical mean chart. When location or dispersion is slightly shifted, limits of $\overline{x} - s^*$ and $\overline{x}^* - s^*$ charts which are small in width and are overlapping and showed same performance. Charts $\overline{x} - s$ and $\overline{x}^* - s$ are wider and overlapping.

## 3.2. Performance evaluation based on average run length ($ARL$)

Performance of a control chart can also be assessed based on Average Run Length. It is defined as the reciprocal of the probability that any point exceeds the control limits. If the process is in-control, the in-control $ARL_0$, should be large and if the process is out-of-control, $ARL_1$ should be very small. The number of subgroups required for the value of the $\bar{x}$ estimator to exceed the control limits was recorded as a run length observation, Run Length. For runs not signaling by the 1000th subgroup, the run length was recorded as 1000. This process is repeated 10,000 times and the results of this simulation study are given in Table 5. The $ARL_0$ was calculated as $ARL_0 = \frac{\sum_{i=1}^{10000} AR_i}{10000}$

Though ARL for the in-controlled data is small due to the effect of censoring it reduces rapidly for out-of-control data. When a small shift occurs $\overline{x}^* - s^*$ chart performs better than classical mean chart.

## 3.3. Performance based on operating characteristic (OC) curve

The ability of the chart to detect process shift in the quality level is the power of the control chart. It is accessed by Operating Characteristic curve which displays probability of Type II error called $\beta$-risk. The OC curve of the $\bar{x}$-chart for Phase II analysis is studied here. The OC function of $\bar{x}$-chart, is the probability $\beta$ of not detecting the shift in the mean on the first
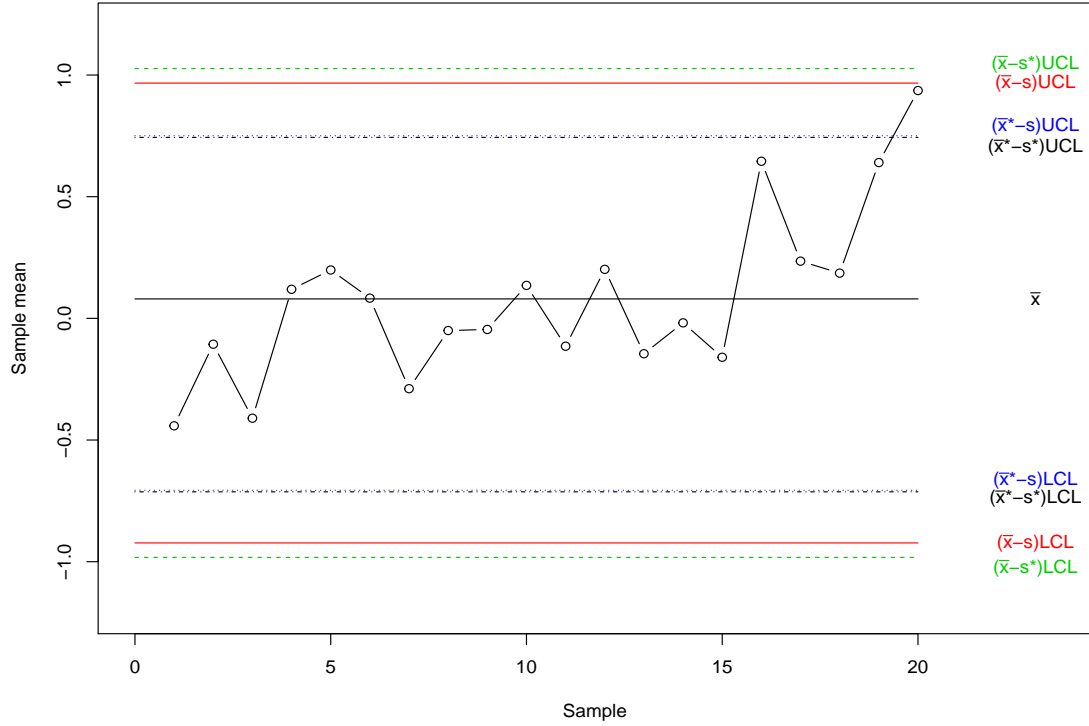
Figure 2: Comparison of $N(2,1)$ contaminated mean control charts, for $n = 10$, $k = 20$ and $r = 1$.

subsequent sample taken after the shift has happened. If the mean in the in-control state is $\mu_0$ and the shift is $\mu_1 = \mu_0 + s\sigma$, $\beta$-risk is defined as

$$\beta - risk = P(LCL \leq \bar{x} \leq UCL | \mu = \mu_1) = \phi(3 - s\sqrt{n}) - \phi(-3 - s\sqrt{n}). \qquad (11)$$

Here $\phi$ is the distribution function of standard normal. To plot the OC curve for the chart, plot the $\beta$-risk against the magnitude of the shift which is expressed in standard deviation units for various sample sizes $n$. Process s.d is considered to be known or to be estimated before considering $\beta$-risk. Smaller probability shows that higher the power of the control chart to correctly detect. Steeper the curves show better probability of detection of shift in the process quality.

Figures 5 and 6 show OC curves of the charts $\bar{x} - s$, $\bar{x} - s^*$, $\bar{x}^* - s$ and $\bar{x}^* - s^*$ for $r = 1$ and $r = 2$ respectively. The smallest $\beta$-risk and bigger slope are observed for $\bar{x} - s^*$ and $\bar{x}^* - s^*$

Table 5: ARL for sample size $n = 10$ and $m = 20$

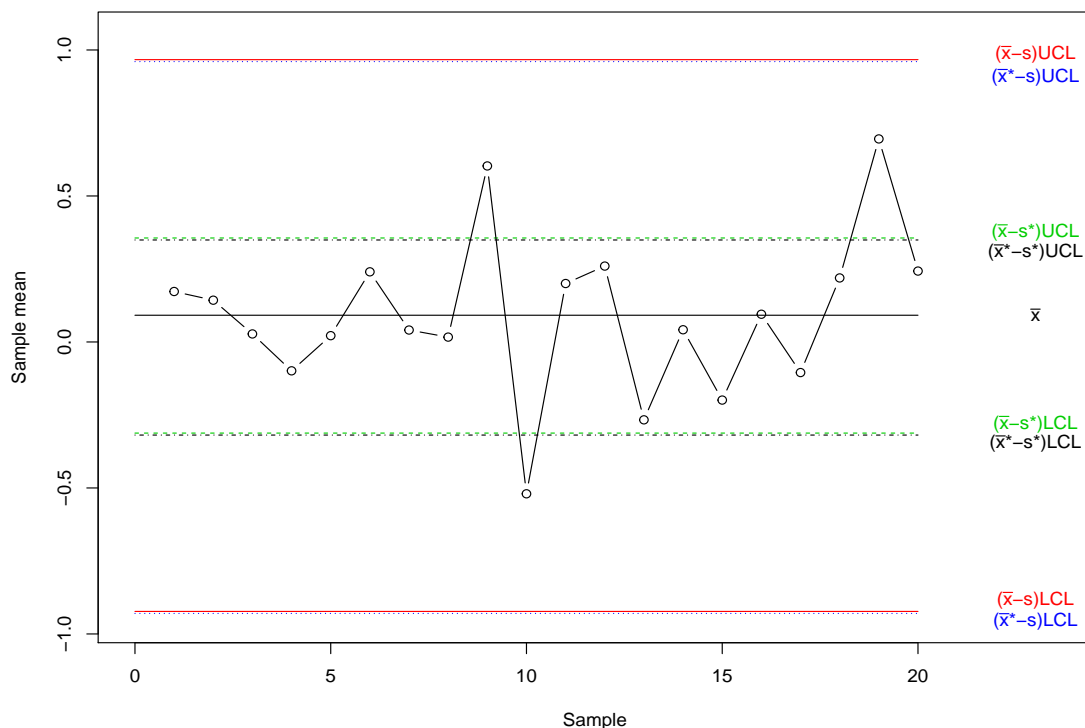| Charts | $\bar{x} - s$ | $\bar{x} - s^*$ | | $\bar{x}^* - s$ | | $\bar{x}^* - s^*$ | |
|---|---|---|---|---|---|---|---|
| Distributions | | $r = 1$ | $r = 2$ | $r = 1$ | $r = 2$ | $r = 1$ | $r = 2$ |
| $N(0,1)$ | 329.043 | 88.397 | 286.470 | 331.862 | 331.624 | 88.563 | 285.471 |
| $N(0,2)$ | 6.2704 | 1.6294 | 0.6712 | 6.2626 | 6.2494 | 1.6328 | 0.6692 |
| $N(1,1)$ | 0.7922 | 0.1186 | 0.024 | 0.7504 | 0.7646 | 0.11 | 0.0218 |
| $N(1,2)$ | 0.9182 | 0.3236 | 0.1426 | 0.8906 | 0.9014 | 0.3152 | 0.142 |
| $N(2,1)$ | 0.0002 | 0 | 0 | 0.0002 | 0.0002 | 0 | 0 |

Figure 3: Comparison of $N(1,1)$ contaminated mean control charts, for $n = 10$, $k = 20$ and $r = 2$.

charts.

# 4. Conclusions

The MMLE has the advantage of censoring extreme observations. When MMLE is applied to industrial process to control process mean, width of the control limits is larger than that of sample mean chart. This happens due to the high weightage given to the remaining extreme observations for compensating the effect of censoring the most extreme values. In order to make this robust estimator suitable to industrial process, authors have made a modification to it. Due to this modification, extreme values get lesser weight while equal weightage is given to all other observations. Corresponding corrections are made to its s.d also. Relative efficiency of this modified estimator for location and its standard error are calculated.

The chart based on MMLE is wider and chart after the modification to MMLE ($\bar{x}^* - s^*$) is smaller than classical mean chart. This small width of modified limits in Phase I analysis is robust to the presence of assignable causes and helps an early detection of assignable causes in Phase II analysis of Shewhart location control charts.

Four different charts say, $\bar{x} - s^*$, $\bar{x}^* - s$ and $\bar{x}^* - s^*$ including classical mean charts for Phase I analysis are considered for the study. When this $\bar{x}^*$ and $s^*$ are used to construct limits, it reduces the width of the limits so that even for small shift an early detection of assignable causes in Phase II analysis is possible. Performance of this modified robust control chart in detecting outliers in presence of contaminated samples is better, and study on ARL and OC also support the results.
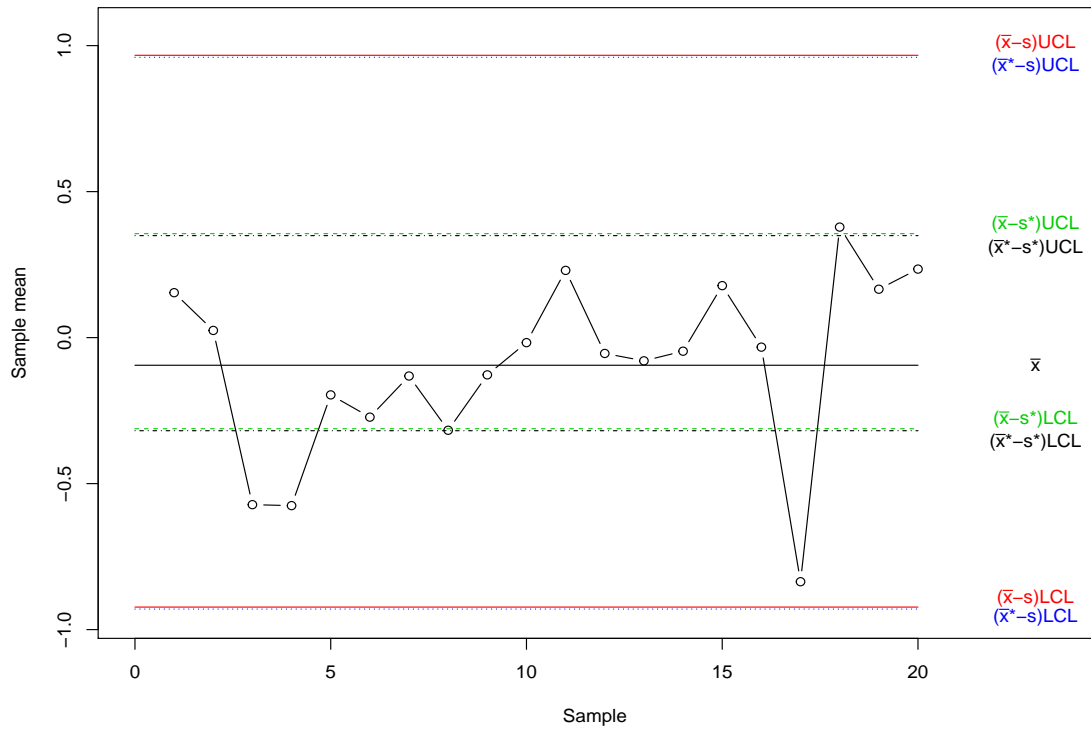
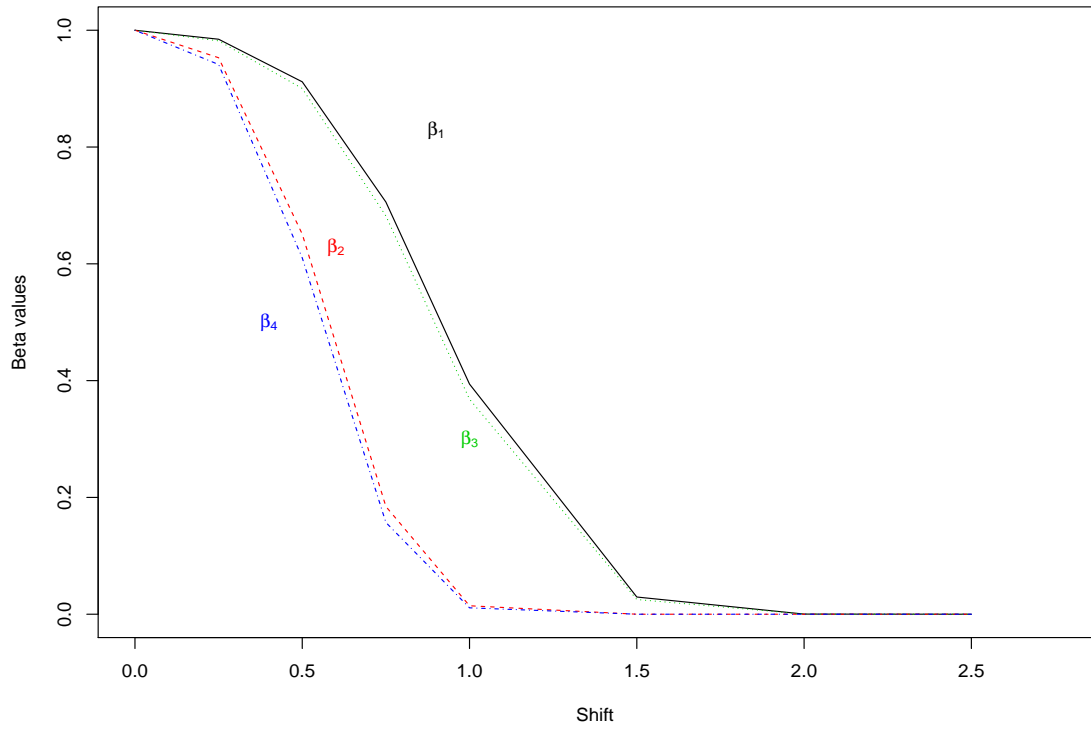Figure 4: Comparison of $N(0, 2)$ contaminated mean control charts, for $n = 10$, $k = 20$ and $r = 2$.



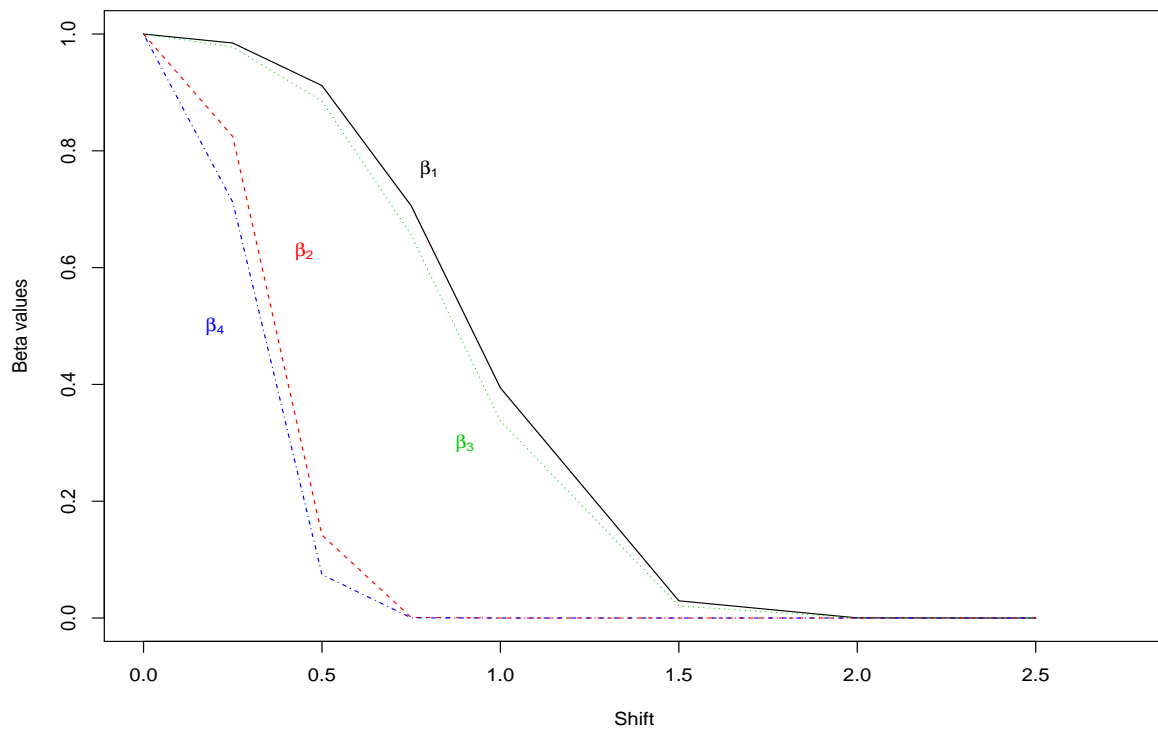Figure 5: Operating curves of the mean charts and MMLE charts $(r = 1)$.

Figure 6: Operating curves of the mean charts and MMLE charts ($r = 2$).

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