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Population age distribution and aggregate saving

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Abstract

The aggregate saving rate results from the interplay of intertemporal preferences across consumption streams over the lifetime of a multitude of agents that compose a population at a given time. Individuals form consumption habits based on their history, age and life expectancy, giving rise to a life cycle consumption pattern sensitive to changes in permanent income. Hence, the aggregate impatience bias towards present vs. future consumption and the aggregate saving rate depend on the evolution of the age distribution of a population. The present paper explores such a relationship by constructing an intertemporal equilibrium model with overlapping generations of finitely lived individuals endowed with recursive preferences that are additively separable over successive consumption streams as in [1].

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1 Introduction

The aggregate saving rate results from the interplay of intertemporal preferences across consumption streams over the lifetime of a multitude of agents that compose a population at a given time. Individuals form consumption habits based on their history, age and life expectancy, giving rise to a life cycle consumption pattern sensitive to changes in permanent income. Hence, the aggregate impatience bias towards present vs. future consumption and the aggregate saving rate depend on the evolution of the age distribution of a population. The present paper explores such a relationship by constructing an intertemporal equilibrium model with overlapping generations of finitely lived individuals endowed with recursive preferences that are additively separable over successive consumption streams as in [1].

2 Continuous time formulation

In the continuous time framework of [1], agents order consumption streams according to the following criterion

$$U({}_0C_\infty) = -\int_0^\infty e^{-\int_0^t u(c)d\tau} dt \tag{1}$$

which allows for additive separability of preferences at any time $T \in [0, \infty]$

$$G(T, C_T, \phi) = -\int_0^T e^{-\int_0^t u(c)d\tau} dt + \phi \cdot e^{-\int_0^T u(c)dt} \tag{2}$$

with aggregate utility from future consumption denoted by

$$\phi = U[{}_TC_\infty] = -\int_T^\infty e^{-\int_T^t u(c)d\tau} dt \tag{3}$$

and additively aggregated to utility from past consumption, upon discounting according to the factor

$$e^{-\int_0^T u(c)dt} \tag{4}$$

In fact, . . . can be rewritten as

$$G(T, C_T, \phi) = -\int_0^T e^{-\int_0^t u(c)d\tau} dt + [-\int_T^\infty e^{-\int_T^t u(c)d\tau} dt] \cdot e^{-\int_0^T u(c)dt} \tag{5}$$

with

$$e^{-\int_0^T u(c)dt} \tag{6}$$

acting like a discount factor over the time interval $[0, T]$. In order to aggregate the utility flow $[-\int_T^\infty e^{-\int_T^t u(c)d\tau} dt]$, it has to be discounted by $e^{-\int_0^T u(c)dt}$.

3 Discrete time formulation

The above formulation can be generalized to a discrete time setting, more suitable to study the quantitative and empirical relationship between the evolution across time of the age distribution of a population and the aggregate impatience and saving rates

$$U({}_1C_2) = -e^{-[u(c(1)])} - e^{-[u(c(1)+u(c(2)])} \tag{7}$$

$$U({}_1C_3) = U({}_1C_2) + [-e^{-[u(c(1)+u(c(2)++u(c(3)])}] \tag{8}$$

$$U({}_1C_2) = -\sum_{t=1}^{2} e^{-\sum_{\tau=1}^{t} u(c(\tau))} \tag{9}$$

3.1 Infinite life span

Agents order consumption streams according to the following criterion

$$U({}_1C_\infty) = -\sum_{t=1}^{\infty} e^{-\sum_{\tau=1}^{t} u(c(\tau))} \tag{10}$$

which allows for additive separability of preferences at any time $T \in [1, \infty]$

$$U({}_1C_\infty) = -\sum_{t=1}^{T-1} e^{-\sum_{\tau=1}^{t} u(c(\tau))} + \phi \cdot \{e^{-\sum_{t=1}^{T-1} u(c(t))}\} \tag{11}$$

with aggregate utility from future consumption denoted by

$$\phi = U({}_TC_\infty) = -\sum_{t=T}^{\infty} e^{-\sum_{\tau=T}^{t} u(c(\tau))} \tag{12}$$

and

$$e^{-\sum_{t=1}^{T-1} u(c(t))} \tag{13}$$

is a discount factor that acts on the utility of consumption streams enjoyed from T to ∞. Such discount factor is distinct from the rate of impatience, and increases with T as longer consumption histories create the habit of discounting more heavily the utility enjoyed from future consumption sequences.

Given the additive structure (11), the marginal utility of a consumption increment at time T is

$$U_T({}_TC_\infty) = u'(c(T)) \cdot \{\sum_{t=T}^{\infty} e^{-\sum_{\tau=T}^{t} u(c(\tau))}\} \tag{14}$$

and its rate of change at time T, along a sequence of constant consumption levels, can be computed as the (negative of) the derivative (with respect to T)

of the logarithm of (14). It is a measure of the pure rate of time preference, denoted by

$$\rho = -\frac{d}{dT}\{log[U_T({}_TC_\infty)]\} = \{\sum_{t=T}^{\infty} e^{-\sum_{\tau=T}^{t} u(c(\tau))}\}^{-1} = -[U_T({}_TC_\infty)]^{-1} \quad (15)$$

Question: Is $u'(c(T)) = 1$ along such a path? The rate of growth of (14) is the derivative (with respect to time) of its logarithm, and such rate is constant.

4 Aggregate impatience

4.1 Finite homogeneous life expectancy

Each generation shares the same finite time horizon, i.e. at birth individuals expect to live for the same finite number of periods, and are endowed with the optimal steady state level of assets[1]. As individuals within a given generation grow older, they generate a longer tail of past consumption levels and expect a shorter sequence of future consumption levels. Eventually, the number of younger generations in existence decreases. Conversely, the greater the number of younger cohorts, the shorter the tails of past consumption, and the lower aggregate impatience. Thus, aggregate impatience grows with time, as generations age, until either a sufficient number of older generations disappear or fertility increases the number of individuals populating younger generations. Each generation in existence at a given time expects a different residual life span.

The (common over a finite set of generations) time horizon at birth is $\overline{T}$.

The distribution of impatience rates at time $T \leq \overline{T} + (n-1)$ and across the $n \in N$ generations can be computed by use of the sequence of future intertemporal utilities.

At time T, generation 1, born at time 1, faces the sequence of future consumption levels from time T to time $\overline{T}$

$$\phi_1(T) = U({}_TC_{\overline{T}}) = \min[1, \max(0, \overline{T} - T)] \cdot [-\sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))}]$$

with $T > \overline{T} \Rightarrow \min[1, \max(0, \overline{T} - T)] = 0$, since at time $T > \overline{T}$ generation 1 has just gone extinct. Generation 2 faces the sequence of future consumption levels from time T to time $\overline{T} + 1$

[1] As will be shown later, the sequence of optimal consumption levels is a function of the sequence of optimal asset levels.

$$\phi_2(T) = U({}_TC_{\overline{T}+1}) = \min[1, \max(0, \overline{T} + (2-1) - T)] \cdot [-\sum_{t=T}^{\overline{T}+1} e^{-\sum_{\tau=T}^{t} u(c(\tau))}]$$

$$= -\sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - \sum_{t=\overline{T}+1}^{\overline{T}+1} e^{-\sum_{\tau=T}^{t} u(c(\tau))}$$

$$= -\sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - e^{-\sum_{\tau=T}^{\overline{T}+1} u(c(\tau))}$$

$$= \phi_1(T) - e^{-\sum_{\tau=T}^{\overline{T}+1} u(c(\tau))}$$

generation 3 faces the sequence of future consumption levels from time T to time $\overline{T}+2$

$$\phi_3(T) = U({}_TC_{\overline{T}+2}) = \min[1, \max(0, \overline{T} + (3-1) - T)] \cdot [-\sum_{t=T}^{\overline{T}+2} e^{-\sum_{\tau=T}^{t} u(c(\tau))}]$$

$$= -\sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - \sum_{t=\overline{T}+1}^{\overline{T}+2} e^{-\sum_{\tau=T}^{t} u(c(\tau))}$$

$$= -\sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - e^{-\sum_{\tau=T}^{\overline{T}+1} u(c(\tau))} - e^{-\sum_{\tau=T}^{\overline{T}+2} u(c(\tau))}$$

$$= -\sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - e^{-\sum_{\tau=T}^{\overline{T}+1} u(c(\tau))}(1 - e^{u(c(\overline{T}+2))})$$

$$= \phi_2(T) - e^{-\sum_{\tau=T+1}^{\overline{T}+2} u(c(\tau))}$$

along a ray with $c(\tau)$ constant. Generation j faces the sequence of future consumption levels from time T to time $\overline{T}+(j-1)$

$$\begin{aligned}
\phi_j(T) = U({}_T C_{\overline{T}+(j-1)}) = \min[1, \max(0, \overline{T}+(j-1)-T)] \cdot [-\sum_{t=T}^{\overline{T}+(j-1)} e^{-\sum_{\tau=T}^{t} u(c(\tau))}] \\
= -\sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))} \\
-e^{-\sum_{\tau=T}^{\overline{T}+1} u(c(\tau))} \\
-e^{-\sum_{\tau=T}^{\overline{T}+2} u(c(\tau))} \\
\vdots \\
-e^{-\sum_{\tau=T}^{\overline{T}+(j-1)} u(c(\tau))} \\
= \phi_{j-1}(T) - e^{-\sum_{\tau=T+(j-2)}^{\overline{T}+(j-1)} u(c(\tau))}
\end{aligned}$$

along a ray with $c(\tau)$ constant, with $\min[1, \max(0, \overline{T}+(j-1)-T)]$ denoting the probability of generation j being alive at time T.

Hence, generation n faces the sequence of future consumption levels from time T to time $\overline{T}+(n-1)$

$$\begin{aligned}
\phi_n(T) = U({}_T C_{\overline{T}+n}) = \min[1, \max(0, \overline{T}+(n-1)-T)] \cdot [-\sum_{t=T}^{\overline{T}+(n-1)} e^{-\sum_{\tau=T}^{t} u(c(\tau))}] \\
= -\sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - e^{-\sum_{\tau=T}^{\overline{T}+1} u(c(\tau))} \\
-e^{-\sum_{\tau=T}^{\overline{T}+2} u(c(\tau))} \\
-e^{-\sum_{\tau=T}^{\overline{T}+3} u(c(\tau))} \\
\vdots \\
-e^{-\sum_{\tau=T}^{\overline{T}+j} u(c(\tau))} \\
\vdots \\
-e^{-\sum_{\tau=T}^{\overline{T}+(n-1)} u(c(\tau))} \\
= \phi_{n-1}(T) - e^{-\sum_{\tau=T+(n-2)}^{\overline{T}+(n-1)} u(c(\tau))}
\end{aligned}$$

The aggregate future intertemporal utility of the population is the aggregation, over the set of generations, of the above sequence. For example, at a time when all generations are alive, $T > n$, and with a population composed of two generations, such aggregate is

$$\Phi^2_{T>n} = \phi_1(T) + \phi_2(T) = -2 \cdot \sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - e^{-\sum_{\tau=\overline{T}}^{\overline{T}+1} u(c(\tau))}$$

while, with three generations it is

$$\Phi^3_{T>n} = \sum_{i=1}^{3} \phi_i(T) = -3 \cdot \sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - 2 \cdot e^{-\sum_{\tau=\overline{T}}^{\overline{T}+1} u(c(\tau))}?$$

$$\Phi^3_{T>n} = \sum_{i=1}^{3} \phi_i(T) = -3 \cdot \sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - 2 \cdot e^{-\sum_{\tau=\overline{T}}^{\overline{T}+1} u(c(\tau))} - e^{-\sum_{\tau=\overline{T}+1}^{\overline{T}+2} u(c(\tau))}$$

With $n \in 1, \ldots, N < \infty$, the aggregate (across the set of generations) expected future utility at a time $T > n$, when all generations are in existence, is

$$\begin{aligned} \Phi^n_{T>n} = \sum_{i=1}^{n} \phi_i(T) = n \cdot \sum_{t=T}^{\overline{T}} e^{-\sum_{\tau=T}^{t} u(c(\tau))} \\ +(n-1) \cdot e^{-\sum_{\tau=\overline{T}}^{\overline{T}+1} u(c(\tau))} \\ +(n-2) \cdot e^{-\sum_{\tau=\overline{T}+1}^{\overline{T}+2} u(c(\tau))} \\ +\ldots \\ +(n-j) \cdot e^{-\sum_{\tau=\overline{T}+1}^{\overline{T}+j} u(c(\tau))} \\ +\ldots \\ +e^{-\sum_{\tau=\overline{T}+1}^{\overline{T}+(n-1)} u(c(\tau))} \end{aligned}$$

where $(n-j),\ j = 1, ..., (n-1)$, takes into account that j out of n generations cannot survive beyond the common time horizon.

As time goes by, generations age and some are gone. As generations age, they stream longer tails of past consumption, thus contributing to the increase of aggregate future intertemporal utility, thus reducing impatience. As they disappear, their past consumption tails no longer add to aggregate future intertemporal utility, increasing impatience. What is the net effect as T extends beyond $\overline{T}$? Aggregate impatience decreases as $T \to n$ as newer generations and ageing cohorts add to aggregate future intertemporal utility. Aggregate impatience monotonically increases as $T > n$ increases past some $T_n < \overline{T} + n$, because too many generations are gone and others get missing.

4.2 An example

In order to analyze the evolution of aggregate impatience of a simple stylized population, three generations are assumed, born at times $1, 2$ and 3 respectively.

The common life expectancy is five periods, $\overline{T} = 5$, so that at times $T = 3, 4, 5$ all generations are alive, while from $T = 6$ to $T = 8$ the number of generations alive, n, converges monotonically to zero. In this case, at time T, generation 1, born at time 1, faces the sequence of future consumption levels from time T to time 5

$$\phi_1(T) = U({}_TC_5) = \min[1, \max(0, 5 + (n-1) - T)] \cdot [-\sum_{t=T}^{5} e^{-\sum_{\tau=T}^{t} u(c(\tau))}]$$

generation 2 faces the sequence of future consumption levels from time T to time 6

$$\phi_2(T) = U({}_TC_6) = -\sum_{t=T}^{6} e^{-\sum_{\tau=T}^{t} u(c(\tau))} = -\sum_{t=T}^{5} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - e^{-\sum_{\tau=T}^{6} u(c(\tau))}$$

and generation 3 faces the sequence of future consumption levels from time T to time 7

$$\phi_3(T) = U({}_TC_7) = -\sum_{t=T}^{7} e^{-\sum_{\tau=T}^{t} u(c(\tau))} = -\sum_{t=T}^{5} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - 2 \cdot e^{-\sum_{\tau=T}^{6} u(c(\tau))} - e^{-\sum_{\tau=T}^{7} u(c(\tau))}$$

Hence

$$\Phi_T^3 = \sum_{i=1}^{3} \phi_i = -3\sum_{t=T}^{5} e^{-\sum_{\tau=T}^{t} u(c(\tau))} - 2 \cdot e^{-\sum_{\tau=T}^{6} u(c(\tau))} - e^{-\sum_{\tau=T}^{7} u(c(\tau))}$$

At time $T = 1$ only generation 1 is alive

$$\Phi_1^3 = -\sum_{t=1}^{5} e^{-\sum_{\tau=1}^{t} u(c(\tau))}$$

At time $T = 2$ only generations 1 and 2 are alive

$$\Phi_2^3 = -2\sum_{t=2}^{5} e^{-\sum_{\tau=2}^{t} u(c(\tau))} - e^{-\sum_{\tau=2}^{6} u(c(\tau))}$$

At times $T = 3, 4, 5$ all generations are alive

$$\begin{aligned}\Phi_3^3 = -3 \sum_{t=3}^{5} e^{-\sum_{\tau=1}^{t} u(c(\tau))} \\ -2 \cdot e^{-\sum_{\tau=3}^{6} u(c(\tau))} \\ -e^{-\sum_{\tau=3}^{7} u(c(\tau))}\end{aligned}$$

$$\begin{aligned}\Phi_4^3 = -3 \sum_{t=4}^{5} e^{-\sum_{\tau=1}^{t} u(c(\tau))} \\ -2 \cdot e^{-\sum_{\tau=4}^{6} u(c(\tau))} \\ -e^{-\sum_{\tau=4}^{7} u(c(\tau))}\end{aligned}$$

$$\begin{aligned}\Phi_5^3 = -3 \sum_{t=5}^{5} e^{-\sum_{\tau=1}^{t} u(c(\tau))} \\ -2 \cdot e^{-\sum_{\tau=5}^{6} u(c(\tau))} \\ -e^{-\sum_{\tau=5}^{7} u(c(\tau))}\end{aligned}$$

At time $T = 6$ generation 1 is no longer in existence, while is the last period for generation 2

$$\begin{aligned}\Phi_6^3 = -2 \cdot e^{-\sum_{\tau=6}^{6} u(c(\tau))} \\ -e^{-\sum_{\tau=6}^{7} u(c(\tau))}\end{aligned}$$

At time $T = 7$ only generation 3 is left

$$\Phi_7^3 = -e^{-\sum_{\tau=7}^{7} u(c(\tau))}$$

4.2.1 Log utility

If $u(c) = \ln(c)$ and consumption is constant and equal to 1 then $u(c) = 0$, the discount factor (13) is constant and equal to 1, so that impatience is only affected by future consumption sequences, and a very simple sequence of future utilities obtains

$$\begin{aligned}\Phi_1^3 = -\sum_{t=1}^{5} e^{-\sum_{\tau=1}^{t} \ln(1)} &= -\sum_{t=1}^{5} e^{0} \\ &= -5 \cdot 1 \\ &= -5\end{aligned}$$

$$
\begin{aligned}
\Phi_2^3 &= -2\sum_{t=2}^{5} e^{-\sum_{\tau=2}^{t}\ln(1)} - e^{-\sum_{\tau=2}^{6}\ln(1)} \\
&= -2\sum_{t=2}^{5} e^{0} - e^{0} \\
&= -2\cdot 4 - 1 \\
&= -9
\end{aligned}
$$

$$
\begin{aligned}
\Phi_3^3 = -3\sum_{t=3}^{5} e^{-\sum_{\tau=1}^{t}\ln(1)} \\
-2\cdot e^{-\sum_{\tau=3}^{6}\ln(1)} \\
-e^{-\sum_{\tau=3}^{7}\ln(1)} \\
= -3\sum_{t=3}^{5} e^{0} \\
-2\cdot e^{0} \\
-e^{0} \\
= -3\cdot 3 \\
-2\cdot 1 \\
-1 \\
= -12
\end{aligned}
$$

$$
\begin{aligned}
\Phi_4^3 = -3\sum_{t=4}^{5} e^{-\sum_{\tau=1}^{t}\ln(1)} \\
-2\cdot e^{-\sum_{\tau=4}^{6}\ln(1)} \\
-e^{-\sum_{\tau=4}^{7}\ln(1)} \\
= -3\sum_{t=4}^{5} e^{0} \\
-3\cdot e^{0} \\
-e^{0} \\
= -3\cdot 2 \\
-2\cdot 1 \\
-1 \\
= -9
\end{aligned}
$$

$$
\begin{aligned}
\Phi_5^3 &= -3\sum_{t=5}^{5} e^{-\sum_{\tau=1}^{t}\ln(1)} \\
&\quad -3\cdot e^{-\sum_{\tau=5}^{6}\ln(1)} \\
&\quad -e^{-\sum_{\tau=5}^{7}\ln(1)} \\
\Phi_5^3 &= -3\sum_{t=5}^{5} e^{0} \\
&\quad -2\cdot e^{0} \\
&\quad -e^{0} \\
&= -3\cdot 1 \\
&\quad -2\cdot 1 \\
&\quad -1 \\
&= -6
\end{aligned}
$$

Aggregate impatience decreases as more generations with longer sequences of future consumption levels come into existence until $T = 3$. From $T = 4$ they become fewer and older, monotonically increasing aggregate impatience to its maximum value in $T = 7$, before the last generation becomes extinct.

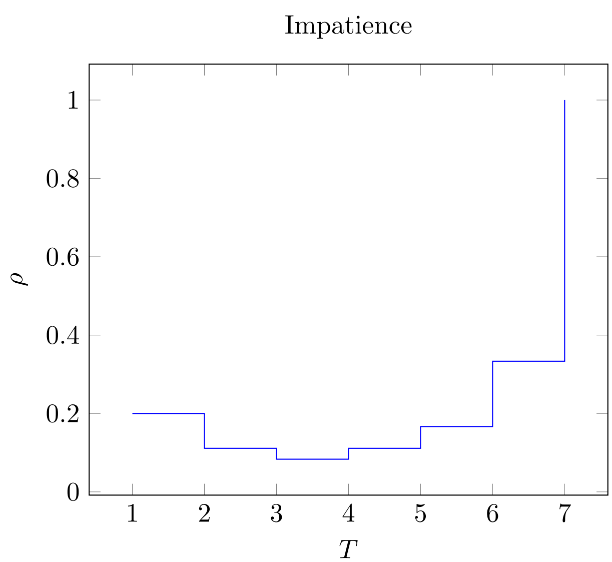

Figure 1: Impatience dynamics with three five periods lived generations

Table 1: Sequence of impatience rates

T	Φ_3^T	ρ
1	-5	0.2
2	-9	0.1111111
3	-12	0.0833333
4	-9	0.1111111
5	-6	0.1666666667
6	-3	0.3333333
7	-1	1

4.2.2 More generations

When the number of generations is extended to $n = 30$, with life expectancy of $\overline{T} = 50$ periods, the population exists over a time span of 80 periods, the following dynamics of aggregate impatience obtains

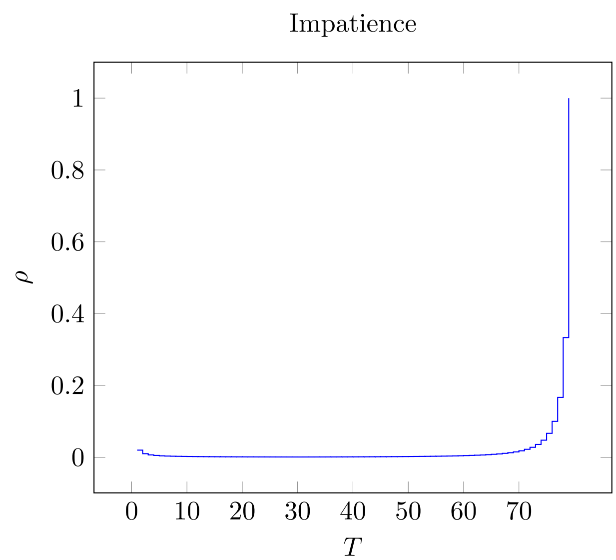

Figure 2: Impatience dynamics with thirty generations

The dynamics of impatience is a negative function of the number of generations alive, with impatience decreasing to a minimum when all generations are in existence and increasing as generations disappear. Thus, a "J" shaped dynamics unfolds with time, as the saving rate increases when younger generations come into existence and impatience eventually increases with population shrinking (because of ageing of finitely lived individuals).

4.3 Heterogeneous life expectancy at birth

Technological progress and economic development may enhance living standards and improve quality of life to the extent of increasing life expectancy at birth for newer generations. Different time horizons imply different discount rates across generations.

5 Conclusions

References

[1] Larry Epstein and J. Allan Hynes. The rate of time preference and dynamic economic analysis. *Journal of Political Economy*, 91:611–35, 1983.