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Money, bonds and the nominal interest rate

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Abstract

This paper studies an economy with trading frictions, and liquid outside bonds in a model *à la* Lagos and Wright [17]. A no-arbitrage condition between nominal assets characterizes coexistence between money and nominal bonds and results in the Fisher equation endogenously determining the equilibrium nominal interest rate.

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1 Introduction

The paper is organized as follows. Section 2 describes the basic framework and the agents' decision problem. Values are characterized in Section 3. Section 4 states the results. The Conclusions end the paper.

2 The environment

The framework of analysis is a modification of LW suited for the characterization of equilibria that feature an endogenous nominal interest rate.

Time is indexed by $t \in \mathbb{N}$.

In each period t two markets open sequentially. The first market to open is decentralized (DM), the second market is centralized (CM).

Agents

There is a closed unit interval $[0, 1]$ of infinitely-lived agents, so that every single agent is a set of zero Lebesgue measure (measure henceforth).

Every agent comes across the whole set of other agents, of which a subset produce and sell appropriate special goods.

It is assumed that the set of special goods appropriate for a particular buyer has α-measure for every agent, with $0 < \alpha \leq 1$.

A setting with an heterogeneous distribution of measures, α_i, of appropriate goods may be used too.

Buyers in the DM are *anonymous*. The exact sense of the term *anonymous* is the subject of some debate...

What is meant by anonymity here is something that rules out trade credit, so that transactions are subject to a *quid pro quo* restriction, which in turn allows money to play a role as a medium of exchange (Kocherlakota [15] and Wallace [24]).

It is assumed that the set of shop owners (sellers) is a subset of $[0, 1]$ with zero measure (e.g the set of shop owners may be assumed to coincide with the Cantor ternary set.)

When an agent comes across a good she demands, the same agent bargains with another agent (the seller) in order to determine the terms of trade.

The terms of trade depend on the distribution of portfolios across shop owners that sell specialized goods (e.g. the distribution of portfolios could be a function on the Cantor ternary set.)

As the mapping from portfolios to agents is bijective, the distribution of portfolios across agents induces a distribution of portfolios across goods for sale.

This distribution is a step function with constant values on each α-measure set of appropriate goods delivered by the same seller.

Special goods

A variety of goods is produced in the DM. Agents specialize in the production of goods other from those they consume, so that autarky cannot be an equilibrium.

Two distinct settings are sketched below that entail the same measure of appropriate sellers and buyers for each agent i.

Specification A.

Assumption A.1) Every agent i specializes in the production of a single good g_i that is demanded by a zero-measure set agents.

For this to be the case, it suffices to impose that each good is demanded by at most two agents even though assuming that each good is demanded by a possibly infinite but still countable set is equivalent.

Accordingly, the measure of people buying the produced good is zero, though the producer may even match a countable infinity of buyers.

As an example, let the set of specialized goods be G. Then, every agent $i \in [0,1]$ demands a variety of goods $G_i \subset G$ with G_i having α-measure. $G_i \cap G_j$ is nonempty with zero measure for every $i \neq j$, i.e. tastes across agents differ almost everywhere, and g_i is not an element of G_i ($\Rightarrow$ no autarky equilibrium.) This implies that *no good is demanded by a set of agents* *with strictly positive measure* and makes the analysis simpler with no loss of generality.

It should be emphasized that this does not imply that if any set of goods are demanded by a nonzero measure set of agents, then such a set of goods must be zero measure.)

The (zero measure) set of buyers appropriate for seller i is denoted by B_i. Hence,

Proposition 1. *a1)* the measure of buyers appropriate for each seller $i \in [0,1]$ is $m(B_i) = 0$, while *a2)* the measure of sellers appropriate for each buyer $i \in [0,1]$ is α.

a1) seems to capture Marx's setting where producers demand money to buy capital, a zero measure set of goods.

Does *a1)* imply that sellers in the DM are elements of a set distinct from the set of buyers (with both selling labor in the CM)? No, as *every agent* both specializes and demands goods.

Summing up, this specification says that

Proposition 1.2. *The measure of purchases of a single agent in the DM is* $\alpha > 0$ *while the measure of a single agent's sales is zero.*

For example, this is the case if most agents (an α-measure set) work only in the CM and buy specialized goods (produced by a zero measure set of agents) in the DM (once a week) and general goods in the CM.

Think of people buying goods from few producers on the internet . . . think of a village fair where lots of people go for a walk and buy from few sellers . . .

Is this like people (it'd be better with like tastes) going to malls, with a very small number of people demanding the same particular good?

As anticipated, portfolios can also be seen as distributed over special goods (in addition to being distributed over agents), in the sense that portfolios are constant over goods that are offered by the same shop owner. So the distribution of portfolios over special goods may be a step function.

Every agent demands an α-measure set of goods, and meets goods rather than sellers. This can be seen as representative of the situation where agents go to huge mall where lots of goods are offered by a relatively little number of shop owners. In such a situation, the chances of a buyer coming across goods she demands are not negligible, while the chances of a particular seller being matched to a particular buyer are considerably fewer.

Specification B.

Assumption B.1) The measure of shop owners is zero.

Assumption B.2) Shop owners (sellers) demand goods from a zero measure set (contrasted with other agents who demand an α-measure set of goods.)

Either B.1 and B.2 jointly, or
Assumption B.3) $m(G_i \cap G_j) = 0$ for every $i \neq j$,
along with
Assumption B.4) $m(G_i) = \alpha$ for every $i \in [0,1]$,
imply:

i) the measure of sellers meeting appropriate goods (in the DM) is zero, and *ii*) the measure of buyers meeting appropriate (in the DM) goods is $\int_{[0,1]} m(G_i) = \int_{[0,1]} \alpha = \alpha$.

Remark. Proposition 1.2 holds in this setting too.

Under the above assumptions, the measure of goods is $\sum\limits_{i\in[0,1]} \alpha = \infty$, while in LW the measure of goods is 1.

This raises the question of *where do those goods that are not produced by any agent come from?* $[\bigcup\limits_{i\in[0,1]} G_i] \sim [0,1]$ is a set of goods that exist in nature (are primitive) and are owned by agents, e.g. different types of labor abilities. *Does it imply that the measure of shop producers-sellers be 1? Is the Cantor ternary set (it has measure 0) an admissible counterexample?*

A convenient side effect of Proposition 1.2 is that value functions are considerably simplified and the constraints on numerical simulations parameterized by α is less stringent (e.g. LW get an upper bound of 0.5 on σ, the money velocity in the DM, which is related to α here).

It turns out that while the value of σ used in simulation by LW made no difference, it does when it comes to simulate the nominal interest rate, i.e. the nominal interest rate is very sensitive to α.

Special goods and assets

Special goods are non-storable and perish at the end of the DM, so that the only assets that can be carried onto the CM are money and bonds. The distribution of money and bonds across agents determines the distribution of agents portfolios. The evolution of this distribution is outlined below.

The evolution of agents portfolios

The distribution of assets holdings F_t changes as a consequence of agents trading.

At every round of decentralized trade, each seller is perfectly happy of selling a good with market price equal to $(1+i)$€ in exchange for a 1€ bond as the bond is about to mature into $(1+i)$€ before any discounting occurs.

An agent entering the DM with (m_t, b_t) exits with $(m_t, b_t)+d_t(\tilde{m}_t, \tilde{b}_t, m_t, b_t)$ in case of a single coincidence sale, and exits with $(m_t, b_t) - d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)$ in case of a single coincidence purchase, with

$$d_t(\tilde{m}_t, \tilde{b}_t, m_t, b_t) = d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t) = (p\nu, p(1-\nu)(1+i_t)^{-1}), \quad \nu \in [0,1] \quad (1)$$

and

$$d : \mathbb{R}_+^4 \times \mathbb{N} \to \mathbb{R}_+^2 \quad (2)$$

As a result, after DM trade agents portfolios are on simplices with extreme points (m_t+p, b_t) and $(m_t, b_t+p(1+i_t)^{-1})$, and $p=0$ for agents who weren't involved in any single coincidence meeting.

This maps F_t to F_t^{DM}.

After bonds mature, agents portfolios are of the type $(m_t + \nu p + (1 - \nu)p + b_t(1+i_t), 0)$, and $p = 0$ in case of no single coincidence meeting.

This maps F_t^{DM} to F_t^b.

Before entering the CM agents allocate the amount of money $m_t + \nu p + (1-\nu)p + b_t(1+i_t)$ between money and bonds, (m_{t_+}, b_{t_+}), to carry onto the CM.

This maps F_t^b to F_{t+}.

Once in the CM an agent with (m_{t_+}, b_{t_+}) chooses $(m', b')(m_{t_+} + b_{t_+}) = (m_{t+1}, b_{t+1})$.

Agents trade assets in the CM in such a way that (m_{t+1}, b_{t+1}) is independent of $(m_{t_+} + b_{t_+})$, i.e. neither the value nor the composition of portfolios matters, and (m', b') is a constant.

Moreover, bonds cannot be exchanged for anything in the CM, so that

$$b_{t+1} = b_{t_+} \tag{3}$$

This maps F_t^b into F_t again.

Preferences in the DM

Agents enjoy utility $u\,(q)$ from q consumption in the DM, where $u'\,(q) > 0$, $u''\,(q) < 0$, $u'\,(0) = \infty$, and $u'\,(\infty) = 0$.

Furthermore, the elasticity of utility $\eta\,(q) = qu'\,(q)\,/u\,(q)$ is bounded by assumption.

Producers incur a utility cost (a disutility) $c\,(q)$ from producing q units of output with $c'\,(q) > 0$ and $c''\,(q) \geq 0$.

Let q^* denote the solution to $u'\,(q^*) = c'\,(q^*)$.

Preferences in the CM

A single good is produced in the CM instead.

In the CM all agents consume and produce, enjoying utility $U\,(x)$ from x units of consumption, with $U'\,(x) > 0$, $U'\,(0) = \infty$, $U'\,(\infty) = 0$ and $U''\,(x) \leq 0$.

The same consumption can be produced from labor by each agent using a linear technology.

This implies that no wealth effects drive demand for money in the CM. Hence, money demand is also independent of trading histories.

Agents discount only between time t-CM and time $t+1$-DM.

This is not restrictive since as in Rocheteau and Wright [21] all that matters is the total discounting between successive periods.

Bonds

As in Zhu and Wallace [25], there are one-period, risk-free assets issued by the government.

In particular, the government uses vending machines to sell bonds in exchange for money. The vending machines feature a record-keeping technology such that they can observe the owner's name and address printed on the certificate.

It is assumed that $b \in \mathbb{R}_+$, so that individuals can invest but not borrow[1].

Agent types and endowments

All bonds automatically turn into money right after decentralized trade has taken place.

Then agents can invest time t-money holdings into the risk-free asset b bearing the (to be determined under equilibrium) gross nominal rate of return $1 + i$.

In principle, exchange of goods for bonds may take place in the CM too. If it does not, then the distribution of bonds across agents is conveniently tractable. So, as a preliminary setup, bonds are are assumed to be illiquid in the CM. This may be the case if, e.g., bonds can be costlessly counterfeit, and counterfeits automatically perish after they change hand.

This in turn requires something like the technology for detecting counterfeits is available in the DM so that sellers do not fear accepting bonds in transactions.

As a result bonds are **endogenously** partially liquid and money is not the only medium of exchange.

Restrictions on bond circulation have been introduced also in [1], [7] and [8]. An exhaustive discussion of illiquid bonds is in [16].

Money supply

A central bank exists that controls the money supply at time t, $M_t > 0$.

[1]Frameworks in which agents can either lend or borrow are in Berentsen, Camera and Waller [4] and Berentsen and Waller [6].

Money supply transforms according to $M_t = \gamma M_{t-1}$. $\gamma \in \mathbb{R}_+$ is a constant and new money is injected, or withdrawn if $\gamma < 1$, through transfers $\pi M_{t-1} = (\gamma - 1) M_{t-1}$ to agents.

Money transfers are lump-sum (i.e. they do not depend on agents' behavior). We restrict attention to policies where $\gamma \geq \beta$, with $\beta \in (0,1)$ denoting the discount factor.

Agents receive lump-sum money transfers π_b at the opening of DM trade. Let $\pi_b M_{t-1} = \pi M_{t-1} / (1-n)$ be the per agent money transfer.

The timing of events is shown in Figure 1.

Stationary equilibria

The analysis is devoted to stationary equilibria characterized by

$$\phi M = \phi_{-1} M_{-1} \tag{4}$$

which implies that $\phi_{-1}/\phi = M/M_{-1} = \gamma$, and aggregate real money balances are constant.

In period t, let $\phi_t = 1/P_t$ denote the real price of money and P_t the price of goods in the CM.

The Fisher equation does not necessarily hold, hence the equivalence of either setting the nominal interest rate or the inflation rate is not granted.

Right after trade of goods has occurred (and before the CM opens), agents have the opportunity to invest their money in nominal bonds.

Interest payments are financed by *lump-sum* taxes levied by the government at the end of decentralized trade.

As interest on bonds is paid right after taxes are collected and before new bonds are sold, interest payments cannot be financed through new bond emissions.

Hence, the government budget constraint expressed in nominal terms is

$$B_t i_t = T_t \tag{5}$$

where B_t is the government debt outstanding at the beginning of time t-CM, and T_t is a lump-sum nominal tax.

Equation (2) states that the interest payment, $B_t i_t$, on bonds is financed by tax revenues. Tax revenues depend on the (to be determined) equilibrium interest rate i_t.

So demand for bonds may affect the amount of taxes that need to be collected, which may disincentive people from investing in bonds. Furthermore, each agent's demand for bonds is zero measure so that nobody is able to exert a non-negligible effect on the amount taxes required to finance interest payment. Hence, no agent can affect the equilibrium nominal interest rate.

3 Values

Aggregate variables enter individual maximization problems as fixed parameters.

Agents decisions are then implied by (common) VFs with money and bond holdings, m and b, as the only arguments.

Let $V(m_t, b_t)$ denote the expected value from trading in the DM with m_t money balances and b_t units of nominal bonds.

Let $W(m_{t_+}, b_{t_+})$ denote the expected value from entering the CM with m_{t_+} units of money and b_{t_+} units of nominal bonds.

It is convenient to sequentially characterize equilibria within a single period starting from the CM.

Centralized market max problems

In the CM agents produce h units of good using h hours of labor, consume x, and adjust their money balances.

Bonds do not mature and cannot be exchanged among agents in the CM.

The real wage per hour is normalized to one.

Discounting explicitly appears in values, Ws, calculated in CMs as they include next period's values, $V_{+1}s$.

No agent accepts bonds as a means of payment in the CM. Hence, if any exchange for bonds were to take place that would have to be a portfolio adjustment between sellers. For ease of analysis this possibility is ruled out. In other words, the distribution of bonds across agents is unaffected by activities taking place in the CM.

3.1 Representative agent's CM problem

There is no dependence of either V or W on t. The notation m_{t+1} stands for money holdings carried onto next DM.

Notice that $\hat{m}_{1,+1}$ denotes $(m_{1,+1}, b_{1,+1})$.

The representative agent's problem at the beginning of the CM

$$W\left(m_{t_+}, b_{t_+}\right) = \max_{x,h,m_{t+1}} \left[U\left(x\right) - h + \beta V\left(m_{t+1}, b_{t+1}\right)\right] \tag{6}$$

such that

$$-h = -x + \phi\left(m_{t_+} - m_{t+1}\right) \tag{7}$$

with $x \in \mathbb{R}_+,\ h, 0 \in H$ a connected closed and bounded subset of $\mathbb{R}_+$, and $m_{t+1} \in \mathbb{R}_+$ denoting the money taken into period $t+1$.

For money demand to be degenerate in the CM, it is sufficient that utility from either labor supply or consumption be linear. Following LW, it is assumed that utility is a linear function of labor supply, h.

Eliminate $-h$ from (6) using (7) and get

$$W\left(m_{t_+}, b_{t_+}\right) = \max_{x,m_{t+1}} \left[U\left(x\right) - x + \phi_t\left(m_{t_+} - m_{t+1}\right) + \beta V\left(m_{t+1}, b_{t+1}\right)\right] \tag{8}$$

or

$$W\left(m_{t_+}, b_{t_+}\right) = \phi_t m_{t_+} + \max_{x,m_{t+1}} \left[U\left(x\right) - x - \phi_t m_{t+1} + \beta V\left(m_{t+1}, b_{t+1}\right)\right] \tag{9}$$

with

$$V(m_{t+1}, b_{t+1}) = V(m_{t+1}, 0) + \alpha u(\tilde{q}(b_{t+1})) \tag{10}$$

where

$$V(m_{t+1}, 0) \tag{11}$$

is utility from buying with money,

The distribution of quantities of appropriate goods purchased with bonds as a medium of payment is assumed to be independent of goods, i.e. it is flat across goods. Denote by $q(g_i, b_{t+1})$ the quantity of good g_i purchased by the agent given her portfolio composition (hence her holdings of bonds).

Depending on m_{t+1}, different levels of b_{t+1} may entail the same distribution of $q(g_i, b_{t+1})$ across G_i. For ease of analysis, the distribution is assumed to depend only on b_{t+1}. Then, total utility gained from exchanges accomplished by use of bonds is given by

$$\int_{G_i} \{u[q(g_i, b_{t+1})]\} = \alpha u[q(g_i, b_{t+1})] \tag{12}$$

Let the constant value of $q(g_i, b_{t+1})$ be denoted by $\tilde{q}(b_{t+1})$. Then

$$\alpha u(\tilde{q}(b_{t+1})) \tag{13}$$

is the additional measure of utility accrued from consuming the quantity $\tilde{q}(b_{t+1})$ purchased across the α-measure of appropriate sellers, using the quantity b_{t+1} of bonds as a medium of payment. The quantity $\tilde{q}(b_{t+1})$ should be equal to $\tilde{q}(b_{t+1}(1+i)€)$, i.e. the quantity purchased using quantity $b_{t+1}(1+i)€$ coins as means of payment. It is worth emphasizing that the quantity $\tilde{q}(b_{t+1})$ is zero measure, and the marginal utility from using an additional unit of bonds as medium of payment is $\alpha u'(\tilde{q})\tilde{q}'(b_{t+1})$, while the marginal utility from using one additional euro coin as means of payment is

Comment. Equilibria with $\tilde{q}(b_{t+1}) : u(\tilde{q}(b_{t+1})) = 1 \Rightarrow \int_{G_i} \{u[q(g_i, b_{t+1})]\} = 1$ could be easier to analyze.

The first order conditions with respect to x and m_{t+1} are

$$U'(x^*) = 1 \tag{14}$$

$$\beta V_{m_{t+1}}(m_{t+1}, b_{t+1}) = \phi_t \tag{15}$$

where the $\beta V_{m_{t+1}}(m_{t+1}, b_{t+1})$ is the seller's marginal benefit of taking money out of the CM and ϕ is its marginal cost.

Equation (14) characterizes the optimal consumption level x^*.

Equation (15) shows that m_{t+1} is independent of b_t and m_t, i.e. the distribution of money holdings across sellers is degenerate at the beginning of the next period because the quasi-linearity assumption on preferences rules out wealth effects on money demand in the CM.

Agents who bring too much cash into the CM spend some buying goods, while those carrying too little sell goods.

Envelopes and no arbitrage

Rather, agents adjust money holdings in the CM so as to exploit arbitrage opportunities as below.

Equations (9) and (3) imply the envelope conditions

$$W_{m_{t_+}} = \phi_t \tag{16}$$

$$W_{b_{t_+}} = \beta V_{b_{t+1}}(m_{t+1}, b_{t+1}) \tag{17}$$

No arbitrage between money and bonds is given by the condition

$$\phi_t = \beta V_{b_{t+1}}(m_{t+1}, b_{t+1}) \tag{18}$$

stating that at the margin no utility gain can be attained from using either medium of exchange.

Notice that (18) is not any agent's decision rule, it is an equilibrium condition that may hold even if $i > 0$.

Combining the efficiency condition (15) of money allocation in the CM with the no arbitrage equation (18), it follows that

$$V_{m_{t+1}}(m_{t+1}, b_{t+1}) = V_{b_{t+1}}(m_{t+1}, b_{t+1}) \tag{19}$$

with

$$V_{b_{t+1}}(m_{t+1}, b_{t+1}) = \alpha u'(\tilde{q})\tilde{q}'(b_{t+1}) \tag{20}$$

by (10).

Hence,

$$V_{m_{t+1}}(m_{t+1}, b_{t+1}) = \alpha u'(\tilde{q})\tilde{q}'(b_{t+1}) \tag{21}$$

i.e. the cost of holding money is equated to the (to be derived) measure of marginal utility from carrying bonds onto next DM.

This reads also as

$$\frac{V_{m_{t+1}}(m_{t+1}, b_{t+1})}{u'(\tilde{q})\tilde{q}'(b_{t+1})} = \alpha \tag{22}$$

i.e. the equilibrium SMS (relative price?) between money and bonds is equal to the measure of goods demanded by agent i in the DM.

In equilibrium, the marginal utility from bonds is greater than that from money because bonds ensure an interest return in addition to acting as a means of payment.

Bonds are more valued than money because if in the next DM I have a bad luck meeting an appropriate seller, bonds make me earn a return anyway that protects me from the inflation tax, while if I am just holding money and can't spend it, then I am bound to pay the inflation tax on all of my wealth.

Remark. $\alpha u'(\tilde{q})\tilde{q}'(b_{t+1})$ is the marginal utility of one additional euro of bonds, and will be determined in Section 3.4.

3.2 DMs

In the DM agents are allowed to barter, exchange specialized goods for money, and exchange specialized goods for bonds.

Let q_b and q_s denote the quantities consumed by a buyer and produced by a seller trading in the DM, respectively.

Agents may not find it optimal to carry entire portfolios to the market as this may reduce bargaining power, but this possibility will not be considered in what follows to simplify the analysis.

Let p be the nominal price of goods in the DM.

As anticipated, an agent carrying the portfolio (m_t, b_t) to the DM exits with $(m_t, b_t) + d_t(\tilde{m}_t, \tilde{b}_t, m_t, b_t)$ in case of a single coincidence sale of the quantity $q_t(\tilde{m}_t, \tilde{b}_t, m_t, b_t)$ to a buyer carrying the portfolio $(\tilde{m}_t, \tilde{b}_t)$, and exits with $(m_t + b_t) - d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)$ in case of a single coincidence purchase of the quantity $q_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)$ from a seller carrying the portfolio $(\tilde{m}_t, \tilde{b}_t)$.

Denote by $V(m_t, b_t)$ the value of entering the DM with portfolio (m_t, b_t).

Then, under either *Specification A* or *Specification B*, each agent $i \in [0,1]$ chooses a portfolio so as to maximize

$$\int_{G_i} \{u[q] + W - d\} + \int_{B_i} \{-\nu[q] + W + d\} \tag{23}$$

Hence, the value function can be written as

$$\begin{aligned}
&V(m_t, b_t) = \\
&\quad \max_{m_t, b_t} \{\alpha \int \{u[q_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)] + W[(m_t + b_t) - d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)]\} dF_t(\tilde{m}_t, \tilde{b}_t) \\
&\quad + m(G_i) \int \{-\nu[q_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)] + W[(m_t + b_t) + d_t(\tilde{m}_t, \tilde{b}_t, m_t, b_t)]\} dF_t(\tilde{m}_t, \tilde{b}_t) \\
&\quad + (1-\alpha) W(m_t, b_t)\}
\end{aligned} \tag{24}$$

where $F_t(\tilde{m}_t, \tilde{b}_t)$ denotes the (induced by sellers specialization) distribution of portfolios across the α-measure set of goods appropriate for agent (buyer) i, and $m(G_i) = 0$ is the measure of buyers appropriate for agent i.

The first term is $\ldots (1-\alpha)$ is the measure $\ldots$

Assume that

$$W[(m_t + b_t) - d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)] = W[(m_t + b_t)] - \phi d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t) \tag{25}$$

and

$$W[(m_t + b_t) + d_t(\tilde{m}_t, \tilde{b}_t, m_t, b_t)] = W[(m_t + b_t)] + \phi d_t(\tilde{m}_t, \tilde{b}_t, m_t, b_t) \tag{26}$$

Then (24) reduces to the simpler form

$$\begin{aligned}
&V(m_t, b_t) = \\
&\quad \max_{m_t, b_t} \{\alpha \int \{u[q_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)] - \phi d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)\} dF_t(\tilde{m}_t, \tilde{b}_t) \\
&\quad + W(m_t, b_t)\}
\end{aligned} \tag{27}$$

LW get rid of this integral because the money the buyer pays is independent of the seller's money holdings (this makes the distribution of money irrelevant but may not be realistic in some cases.)

In other words, the zero measure of sellers jointly with the independence of the money payment on the seller's money holdings (and, different from LW, no chances of bartering) give the following value

$$\begin{aligned} V(m_t, b_t) = \\ &\max_{m_t, b_t}\{\alpha\{u[q_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)] - \phi d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)\} \\ &+ W(m_t, b_t)\} \end{aligned} \tag{28}$$

3.2.1 Bargaining

In the Nash problem

$$\max_{q, d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t), d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)} [u(q) - \phi d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)]^{\theta}[-\nu(q) + \phi d_t(\tilde{m}_t, \tilde{b}_t, m_t, b_t)]^{1-\theta} \tag{29}$$

what the buyer pays is equal to what the seller gets, so that

$$\max_{q, d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)} [u(q) - \phi d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)]^{\theta}[-\nu(q) + \phi d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)]^{1-\theta} \tag{30}$$

Assume $\theta = 1$ so that the above problem simplifies to

$$\max_{q, d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)} [u(q) - \phi d_t(m_t, b_t, \tilde{m}_t, \tilde{b}_t)] \tag{31}$$

The solution depends both on the degree of mildness of disutility from labor and on whether the budget constraint binds.

If disutility is mild enough and the constraint binds the solution is characterized by

$$u'(q) > 0, \lambda_b > 0 \tag{32}$$

with λ_b denoting the Lagrange multiplier on the buyer's budget constraint, and trade is inefficient. Otherwise, $u'(q)$ must equate the utility cost of giving money up.

Is this expressed by

$$u'(q) = \phi \tag{33}$$

?

If the answer is yes, assume $u(q) = \ln(q)$ so that

$$q^* = \phi^{-1} \tag{34}$$

so that the terms of trade are (ϕ^{-1}, ϕ), i.e. the quantity ϕ^{-1} is exchanged at the utility price ϕ of a unit of money (equivalently, for a unit of money).

If there is no disutility from labor, then the quantity produced and exchanged is efficient, and so $d = (m^*, b^*) = (m_t, b_t)$, where (m^*, b^*) is the least amount of assets (money and bonds) sufficient to induce the seller to produce and offer the quantity q^*.

If the answer is no, consumer equilibrium is given by

$$u'(q) = \phi p \tag{35}$$

Assuming $p = 1$, it follows that

$$q^* = \phi^{-1} = 1 \tag{36}$$

and the terms of trade reduce to $(q^*, p) = (1, 1)$, i.e. the quantity 1 is exchanged at the utility price 1 of a unit of money (equivalently, for a unit of money).

Again, if there is no disutility from labor, then the quantity produced and exchanged is efficient, and so $d = (m^*, b^*) = (m_t, b_t)$, where (m^*, b^*) is the least amount of assets (money and bonds) sufficient to induce the seller to produce and offer the quantity $q^* = 1$.

The value (28) becomes

$$V(m_t, b_t) = \alpha(\ln[1] - 1) + W(m_{t_+}, b_{t_+}) \tag{37}$$

or

$$V(m_t, b_t) = -\alpha + W(m_{t_+}, b_{t_+}) \tag{38}$$

Assume linear utility in the CM, $U(x) = x$, and use (8) to get

$$V(m_t, b_t) = -\alpha + \max_{(m_{t+1}, b_{t+1})} \{\phi_t m_{t_+} - \phi_t m_{t+1} + \beta V(m_{t+1}, b_{t+1})\} \tag{39}$$

where $\phi(m_{t_+} - m_{t+1})$ is the value of money inside next period portfolio net of the cost of acquiring it, and (3) implies maximization only w.r.t. m_{t+1}.

The term $\nu_0(s)$ in LW2002 is zero hero.

Hence,

$$V(m_t, b_t) = -\alpha + \max_{m_{t+1}} \{\phi_t m_{t_+} - \phi_t m_{t+1} + \beta V(m_{t+1}, b_{t+1})\} \tag{40}$$

Repeated substitution gives

$$V(m_t, b_t) = -\alpha + \phi_t m_{t_+} + \sum_{j=t}^{\infty} \beta^{j-t} \max_{m_{j+1}} \{-\phi_j m_{j+1} + \beta[\nu_{j+1}(m_{j+1}, b_{j+1}) + \phi_{j+1} m_{j+1}]\} \tag{41}$$

where $\nu_{j+1}(m_{j+1}, b_{j+1})$ depends on b_{j+1} because of the medium of exchange role played by bonds carried onto next DM.

or

$$V(m_t, b_t) = -\alpha + \phi_t m_{t_+} + \sum_{j=t}^{\infty} \beta^{j-t} \max_{m_{j+1}} \{m_{j+1}(\beta\phi_{j+1} - \phi_j\} \tag{42}$$

This is simpler than LW as we got rid of their ν_{t+1} which depended on F_{t+1}, $\nu_{t+1}(F_{t+1})$. So there is no dependence of the sequence of m_j's on F_{t+1} (LW say it only influences the intercept of the VF and not the m_{t+1}).

$V(m_t, b_t)$ is linear in m_{j+1}. If $\beta\phi_{j+1} - \phi_j > 0$, i.e. $\beta\phi_{j+1} > \phi_j$ or $\frac{\phi_{j+1}}{\phi_j} > \frac{1}{\beta}$, then there is no solution to the problem of choosing m_{t+1}... why? Is the no-arbitrage condition of help?

Looks like equilibrium requires $\beta\phi_{j+1} < \phi_j$ as in LW. Does it imply that the optimal m_{t+1} is zero and agents only hold bonds? Yes it implies $m^*_{t+1} = 0$. Don't know if bonds are positive.

Notice that LW characterize monetary equilibrium by any path for $\{q_{t+1}\}$ satisfying $m_{t+1} < m^*_{t+1}$ (LW, p. 472).

If $\phi_{j+1} = \phi_j$ as, given b_{t_+}, eq. (21) suggests (this holds also because of Lemma 3 in LW2002) …then $\beta\phi_{j+1} < \phi_j$ holds and $m^*_{t+1} = 0$. If so, then $V(m_t, b_t) = -\alpha + \phi_t m_{t_+}$.

Whose hands the money spent in the CM goes? As nobody is carrying any money into next DM, everybody must be spending the whole of money holdings (including money injection which takes place at the beginning of the CM).

Hence, agents find it optimal to maximize the value of their portfolios and the portfolio constraint always binds. As a consequence, neither the equilibrium demand for money nor for bonds can be zero. Hence, money and bonds coexist? Even though agents try to substitute money for bonds …

…

General bargaining solution

The general solution characterized by LW consists of the seller spending the $d_t(m, \tilde{m}) = \min(m_t, m^*)$.

If $m_t = \min(m_t, m^*)$ then the buyer gets $q_t(m, \tilde{m}) = \tilde{q}_t(m) \leq q^*$.

If $m^* = \min(m_t, m^*)$ the cash constraint is not binding and the buyer gets $q_t(m, \tilde{m}) = q^*$ (and eventually disposes or what of excess money holdings?)

(This consists of either the seller exchanging all of his money holdings ($d_t(m, \tilde{m}) = m$) for a quantity weakly less than the efficient level $q_t(m, \tilde{m}) = \tilde{q}_t(m) \leq q^*$ (if $m_t \leq m^*$ with m^* denoting the least amount of money sufficient to buy q^*), or the buyer giving all of her money holdings ($d_t(m, \tilde{m}) = m < m^*_t$) up for a lesser quantity ($m_t < m^*$ and the budget constraint is binding.))

Hence, in LW the solution to the bargaining problem only depends on the buyer's money holdings m_t and I cannot get any discount from a starving seller!

Remark. The LW solution to the bargaining problem $\Rightarrow$ the buyer cannot get any utility from money in excess of $m^* \Rightarrow \nu'_{t+1}(m_{t+1}) = 0$ for all $m_{t+1} \geq m^*_{t+1}$. $\Rightarrow$ any equilibrium must satisfy $\phi_t \geq \beta\phi_{t+1}$.

If bonds allow buyers to get utility from that extra cash, will the minimum inflation rate consistent with equilibrium still be the Freidman rule? see LW p. 471.

Let agents with more than m^* lend to those with less. This should be welfare improving as more people consume closer to (at) the efficient level. There is a role for credit with no banks, no government, and no ex-post heterogeneity!

Is $m^*_{t+1} = m^*$?

The bargaining solution can be used to simplify the value function (24).

...

3.3 Compare monetary bargaining to monetary plus bonds bargaining

Portfolio composition matters

Imagine sellers prefer money, but may consider accepting complementary (in addition to money) assets in exchange for goods in case the buyer money holdings are short of m^*.

In this case, different from LR, is not the total value of portfolios that matters, but the composition too is of importance.

What if $m_t = \min(m_t, m^*)$ and $m^* \leq m_t + b_t(1+i_t)^{-1}$, i.e. $m^* = \min(m_t + b_t(1+i_t)^{-1}, m^*)$?

In this case, the seller has an incentive to accept bonds at a discount and the buyer has an incentive to use bonds to pay for the efficient quantity.

For instance, it may either happen that the seller gets $\min(m_t + b_t(1+i_t)^{-1}, m^*)$ for q^*, or that the buyer gets q^* for may get $\min(m_t + b_t(1+i_t)^{-1} - \epsilon, m^*)$.

Does subsidization of bond payments emerge? ...

3.4 The equilibrium nominal interest rate

Lagos and Rocheteau give agents periodic access to asset markets so as to allow for competition between real and nominal assets. The present setting departs from theirs in that competition is allowed for between nominal assets (rather than real assets) as means of payment (instead of stores of value.)

Portfolio composition does not matter

As sellers are indifferent between 1€ cash and $(1+i)^{-1}$ € bond, the buyer fully appropriates for the returns exchange using bonds provides. Such returns equal the utility accrued from investing $(1+i)^{-1}$€ coins in bonds and transforming the bonds back to 1€ cash (which includes the interest payment) upon bargaining. Hence, the quantity used to evaluate such returns is the quantity resulting from the bargaining solution (equation (36).) Then, utility from buying in the next DM using a 1€ bond as medium of payment must be

$$\alpha u'(\tilde{q})\tilde{q}'(b_{t+1}) = \phi(1+i)(1+\pi)^{-1}u'(q(b)) \tag{43}$$

as the seller evaluates this as the same as $(1+i)$€ coins.

In particular, $(1+i)$€ is the (monetary) before inflation tax return of 1€ coin invested in bonds.

$(1+i)(1+\pi)^{-1}$€ (π is a pure number) is the (monetary) return of 1€ coin invested in bonds net of the inflation tax.

$\phi(1+i)(1+\pi)^{-1}$ (the unit of measure of ϕ is €) is the real (purchasing-power, i.e. in terms of maximum amount of goods that can be purchased in the next DM) return of 1€ coin invested in bonds.

$\phi(1+i)(1+\pi)^{-1}u'(q(b))$ is the return of 1€ coin invested in bonds measured in utils.

Such return is affected by monetary policy as the equality $\gamma = (1+\pi)$ holds in equilibrium.

Hence, equation (22) gives

$$\frac{V_{m_{t+1}}(m_{t+1}, b_{t+1})}{\phi(1+i)(1+\pi)^{-1}u'(q(b))} = 1 \tag{44}$$

which, using equation (15), becomes

$$\frac{\phi\beta^{-1}}{\phi(1+i)(1+\pi)^{-1}u'(q(b))} = 1 \tag{45}$$

or

$$(1+i) = \frac{\phi\beta^{-1}}{\phi u'(q(b))} \tag{46}$$

The trading solution (36) gives

$$(1+i) = \phi\beta^{-1}(1+\pi) \tag{47}$$

or

$$(1+i) = \frac{(1+\pi)}{\beta} \tag{48}$$

or

$$i_t = \frac{(1+\pi)}{\beta} - 1 \tag{49}$$

which is the equilibrium nominal interest rate.

If every meeting is a single coincidence, then

$$i_t = \frac{(1+\pi)}{\beta} - 1 \tag{50}$$

Taking account of $(1 + i_R) = \beta^{-1}$, this can be rewritten as

$$(1+i_t) = (1+\pi)(1+i_R) \tag{51}$$

which is the Fisher equation.

So that the Fisher equation holds in an economy with nominal bonds and no (frictions due to) absence of single coincidence.

If instead I use $c(q) = q$ so that $u'(q(b)) = c'(q(b)) = 1$, the same result obtains.

In general, using $\beta = \frac{1}{1+i_R}$,

$$(1 + i_t) = (1 + \pi)(1 + i_R) \tag{52}$$

and

$$i_t = (1 + \pi)(1 + i_R) - 1 \tag{53}$$

follow.

In words, the equilibrium nominal interest rate exceeds (is short of) the values predicted by the Fisher equation if every agent demands a set of goods with measure less (more) than 1. The equilibrium interest rate is equal to the values from the Fisher equation *iff* the measure of goods demanded by every agent is equal to 1, which in the present setting is also the measure of agents.

In particular, the nominal interest rate is negative when $\alpha > (1 + \pi)(1 + i_R)$, which requires either a deflation or a set of appropriate sellers with measure (α) larger than the set of agents (which is 1, but may be zero if agents are elements of the Cantor ternary set.) Is there any evidence of negative nominal interest rates, positive values from the Fisher equation notwithstanding? Such cases should encompass examples with deflation as, according to the Fisher equation, negative nominal rates should be caused by deflation. I.e., unless I have deflation, I cannot have negative nominal interest rates if the Fisher equation holds.

Moreover, here negative interest rates are possible under no arbitrage equilibrium (see [2], p. 2).

"Indeed, negative nominal rates are deemed impossible in equilibrium because they afford an arbitrage, and the absence of arbitrage is normally considered an equilibrium condition that is required for market clearing."([2]).

What is this arbitrage? Here negative nominal interest rates may emerge under no arbitrage, which is the condition that determines the endogenous equilibrium nominal interest rate.

Do negative rates act (as is the case in [2]) as a 100% tax on nominal wealth (i.e. bonds plus cash?)?

Probably, the uniqueness of equilibrium proven by Wright (if the result extends here) rules out any problem of *strict implementation*, i.e. adoption of policy rules that avoid the existence of equilibria different from the desired one.

How comes the government is paid as a borrower (the nominal interest rate is negative)? Bargaining involves the purchase of a good out of an α-measure set of goods. When α is sufficiently larger than 1, the variety of goods outnumbers the number of agents by far. The chances of sellers selling a sufficiently large measure of goods are little. This captures the idea of an "insufficient equilibrium-aggregate demand". Hence, sellers may be willing to subsidize agents' purchases (this may apply in particular to primitive and perishable goods) as a way of minimizing losses (this is somehow related to taxation of inelastically supplied labor). Subsidization may be allowed for by sellers accepting bonds as a means of payment, with subsidization in the form of a premium on bond payments, which is equivalent to discounting bonds at a negative nominal interest rate.

4 Quantitative analysis

In LW $i_R = 0.04$, take inflation data for π and compute series for i_t parameterized by $\alpha \cong 1$ and according to eq. (53).

How do I choose α? Maybe I should try to find the α that best fits the relationship between the inflation rate and the nominal interest rate.

A simple way of estimating α consists of taking the data with monthly frequency, take the monthly real interest rate, and calculate the monthly α's. Then make an average and take it as an estimate.

Take α as a deep parameter and use the estimated value to check if it is consistent with negative nominal rates when they occurred. They observed rates should be negative when $\pi \leq -i_R$ because the perturbed Fisher equation gives this as a sufficient condition.

Best fit in what sense? Lucas (2002) fits are choices among members of a parametric family (i.e. a finite number) of functions.

Remarks on graph 2. I guess that the variability in `F est` cannot be present in the `CPN3M` because that is the interest rate in what are contracts not indexed on inflation, and so it is less sensitive to monthly changes on yearly inflation.

Is a moving average of `NF est` more appropriate? Of course not, as the equilibrium nominal interest rate has to compensate agents for the inflation tax.

`F est` compensates as well, but it overpredicts values by a large amount all over the time period considered.

The `NF est` fit is very good from around July 2000 until July 2002. Its global overprediction is way less.

Check if this is a result of the FED pegging the `TB` to `CPI`.

`CPN3M` seems to have not been incorporating inflation until around August 2007. If this is correct, then the volatility of `CPI` seems to be a cost for those who have been lending money through `CPN3M` contracts (and a gain from January 1997 until around July 2000, a period during which inflation displayed less volatility).

FED cuts of `TB` are likely to have resulted in subsidization of indebted agents as, e.g., those who are paying mortgages, and of public debt servicing.

Is there any instance when equilibrium nominal rates are negative? It would be worth trying to estimate Japan deflation and “liquidity trap”.

5 Conclusions

To be added.

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